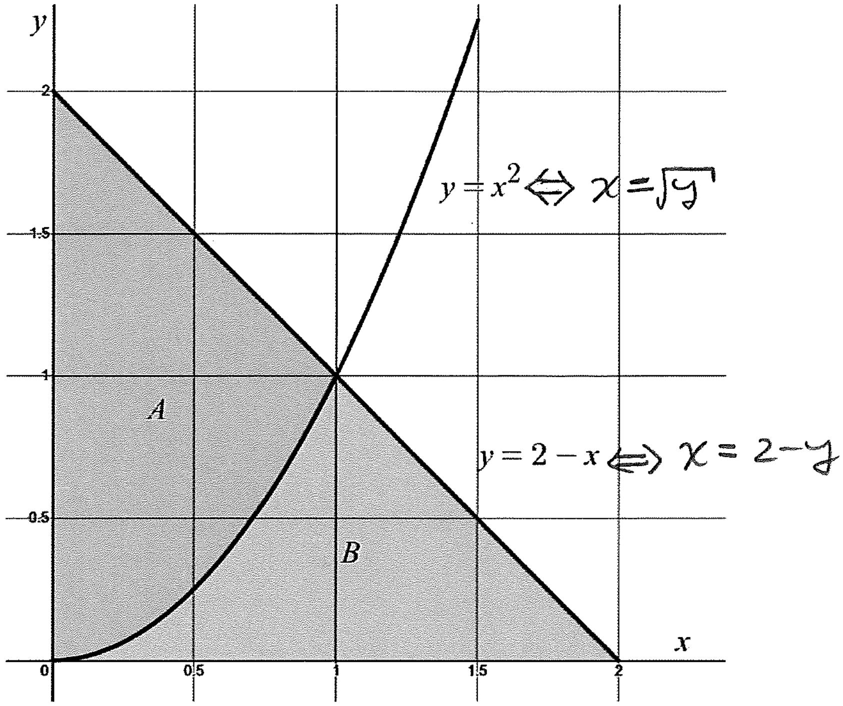
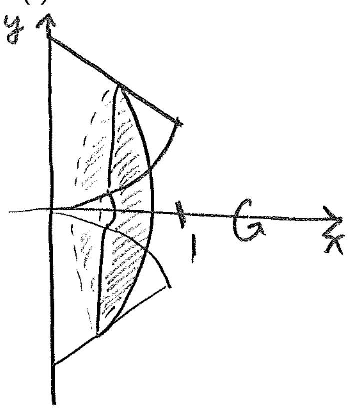
Question 1: For this question use the following figure:



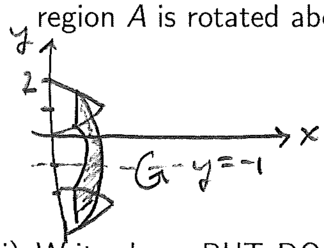
(i) Determine the volume of the solid produced when region A is rotated about the x-axis.



$$V=\int_0^1 \pi[(2-x)^2-(x^2)^2]\,dx$$
$$=\int_0^1 \pi\,[4-4x+x^2-x^4]\,dx$$
$$=\pi\left[4x-\frac{4}{2}x^2+\frac{x^3}{3}-\frac{x^5}{5}\right]_0^1$$
$$=\pi\left[4-2+\frac{1}{3}-\frac{1}{5}\right]$$
$$=\pi\left[\frac{30+5-3}{15}\right]=\boxed{\frac{32\pi}{15}}$$

[5]

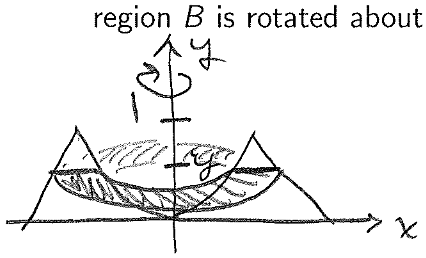
(ii) Write down BUT DO NOT EVALUATE a definite integral for the volume of the solid generated when region A is rotated about the line $y=-1$.



$$V=\int_0^1 \pi\left[(2-x-(-1))^2-(x^2-(-1))^2\right]dx$$
$$=\pi\int_0^1\left[(3-x)^2-(x^2+1)^2\right]dx$$

[3]

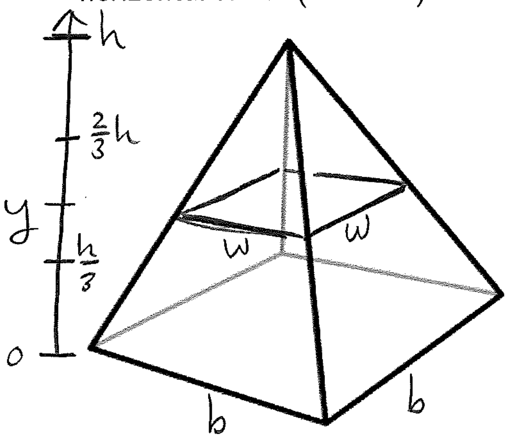
(iii) Write down BUT DO NOT EVALUATE a definite integral for the volume of the solid generated when region B is rotated about the y-axis.

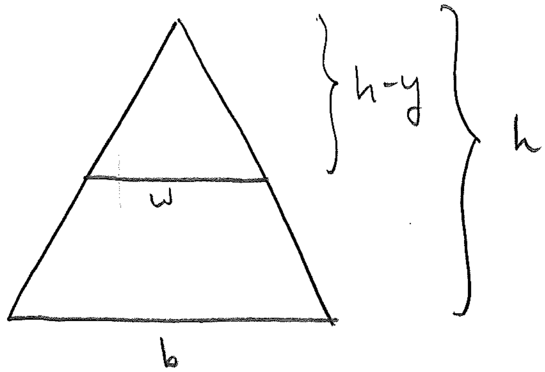


$$V=\int_{y=0}^{1}\pi\left[(2-y)^2-(\sqrt{y})^2\right]dy$$

[2]

Question 2: A pyramid has height h and a square base of side length b. The pyramid is divided into three horizontal levels (or floors) each of height $h/3$. Use integration to determine the volume of the middle level.





$$\frac{h-y}{w} = \frac{h}{b}$$

$$w = \frac{b}{h}(h-y).$$

$$\therefore A(y) = w^2 = \frac{b^2}{h^2}(h-y)^2.$$

$$V = \int_{y=\frac{h}{3}}^{\frac{2}{3}h} \frac{b^2}{h^2}(h-y)^2\,dy$$

$$= \frac{b^2}{h^2}\left[\frac{(h-y)^3}{-3}\right]_{\frac{h}{3}}^{\frac{2h}{3}}$$

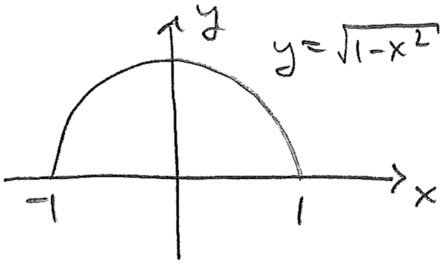
$$= \frac{b^2}{-3h^2}\left[\left(\tfrac{1}{3}h\right)^3 - \left(\tfrac{2}{3}h\right)^3\right]$$

$$= \frac{-b^2}{3h^2}\left[-\frac{7h^3}{27}\right]$$

$$= \boxed{\frac{7b^2h}{81}}$$

[5]

Question 3: Recall that the graph of $y = \sqrt{1-x^2}$ is the top half of a circle of radius 1 and center $(0,0)$. Use integration (the arc length formula) to find the length of this curve. (We know what the answer should be; use integration to show that you get the correct result.)

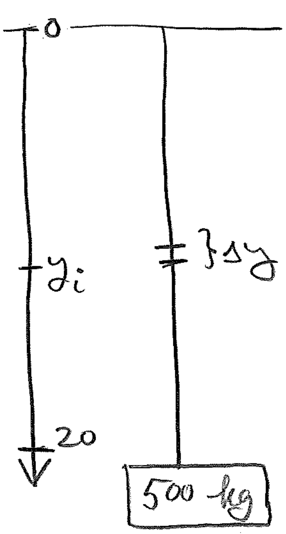


$$y' = \frac{-2x}{2\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$$

$$L = \int_{-1}^{1} \sqrt{1+(y')^2}\,dx$$
$$= 2\int_0^1 \sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\,dx$$
$$= 2\int_0^1 \sqrt{1+\frac{x^2}{1-x^2}}\,dx$$
$$= 2\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$$
$$= 2\left[\arcsin(x)\right]_0^1$$
$$= 2\left(\frac{\pi}{2}-0\right) = \boxed{\pi}$$

[5]

Question 4: A building elevator system has car of mass 500 kg and a steel cable of linear density 2 kg/m attached to it. When the elevator car is called to the top floor the electric motor pulling the cable shortens it from 20 m to 0 m. How much work did the motor do? Recall that acceleration due to gravity is $g = 9.8$ m/s^2, however you may leave the constant g in your final answer .

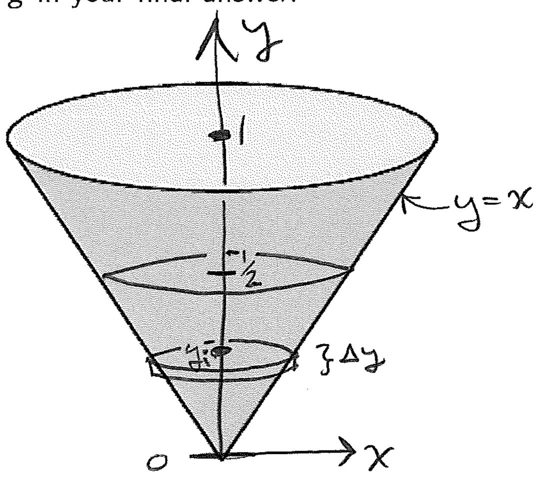


$$W_{Total} = W_{car} + W_{cable}$$
$$W_{car} = (\text{weight})(\text{distance})$$
$$= (500)(g)(20)$$
$$= 10000g$$
$$W_{cable} = \lim_{n\to\infty} \sum_{i=1}^{n} 2g y_i \Delta y$$
$$= \int_0^{20} 2g y\,dy$$
$$= g\left[y^2\right]_0^{20}$$
$$= 400g$$

$$\therefore W_{total} = 10000g + 400g$$
$$= \boxed{10400g \text{ J}}$$

[5]

Question 5: The line $y = x$, $0 \leq x \leq 1$, is rotated about the y-axis to form a cone-shaped vessel which is then filled with water to a depth of 1/2 m. (Here the units for x and y are in meters.) Find the work required to empty the vessel by pumping all of the water to the top of the tank. Recall the density of water is $\rho = 1000$ kg/m^3 and acceleration due to gravity is $g = 9.8$ m/s^2, however you may leave the constants ρ and g in your final answer.



Disk at y_i has radius $x_i = y_i$, so has volume $\pi(y_i)^2 \Delta y$.

∴ mass of disk is $\rho \pi y_i^2 \Delta y$

∴ weight of disk is $\rho g \pi y_i^2 \Delta y$

∴ Work to lift disk to 1 m level is $\rho g \pi y_i^2 (1-y_i)\Delta y$

∴ Total work is

$$W \approx \sum_{i=1}^{n} \rho g \pi y_i^2 (1-y_i)\Delta y$$

Letting $n \to \infty$:

$$W = \int_0^{\frac{1}{2}} \rho g \pi y^2 (1-y)\,dy = \rho g \pi \left[\frac{y^3}{3} - \frac{y^4}{4}\right]_0^{\frac{1}{2}} = \rho g \pi \left[\frac{1}{24} - \frac{1}{64}\right] = \boxed{\frac{5}{192}\rho g \pi \text{ J}}$$

[5]

Question 6: Solve the differential equation with given initial condition. State your final answer in explicit form (that is, isolate y in your final answer.)

$$y' \tan(x) = \sqrt{3} + y, \quad y(\pi/3) = \sqrt{3}$$

$$\frac{dy}{dx}\,\frac{\sin(x)}{\cos(x)} = \sqrt{3} + y$$

$$\int \frac{1}{\sqrt{3}+y}\,dy = \int \frac{\cos(x)}{\sin(x)}\,dx$$

$u = \sin(x)$
$du = \cos(x)\,dx$

$$\ln|\sqrt{3}+y| = \ln|\sin(x)| + C_1$$

$$|\sqrt{3}+y| = C_2\,|\sin(x)|$$

$$\sqrt{3}+y = C_3 \cdot \sin(x)$$

$$y = C_3 \cdot \sin(x) - \sqrt{3}$$

$y(\frac{\pi}{3}) = \sqrt{3}$, so

$$\sqrt{3} = C_3 \cdot \sin\left(\frac{\pi}{3}\right) - \sqrt{3}$$

$$\sqrt{3} = C_3 \cdot \frac{\sqrt{3}}{2} - \sqrt{3}$$

$$C_3 = 2\sqrt{3} \cdot \frac{2}{\sqrt{3}} = 4$$

$$\boxed{\therefore\ y = 4\sin(x) - \sqrt{3}}$$

[5]

Question 7: Determine the limit of the sequence with terms $a_n = \sqrt{\dfrac{n+1}{9n+1}}$, $n = 1, 2, 3, \ldots$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{\frac{n+1}{9n+1}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{1+\frac{1}{n}}{9+\frac{1}{n}}}$$

$$= \sqrt{\frac{1}{9}}$$

$$= \boxed{\frac{1}{3}}$$

[3]

Question 8: Write out the first three terms of the geometric series $\sum_{n=0}^{\infty} \dfrac{\pi^n}{5^{n+1}}$ and then decide if it converges. If it does converge then state the sum.

$$\sum_{n=0}^{\infty} \frac{\pi^n}{5^{n+1}}$$

$$= \frac{1}{5} + \frac{1}{5}\cdot\left(\frac{\pi}{5}\right) + \frac{1}{5}\cdot\left(\frac{\pi}{5}\right)^2 + \cdots$$

Geometric, $a = \frac{1}{5}$, $r = \frac{\pi}{5}$

$|r| < 1$, so series converges

to $\dfrac{a}{1-r} = \dfrac{(1/5)}{1-(\frac{\pi}{5})} = \boxed{\dfrac{1}{5-\pi}}$

[4]

Question 9: Does the series $\sum_{n=0}^{\infty} \dfrac{e^n}{n^2}$ converge? Explain.

No. $\displaystyle\lim_{n\to\infty} \frac{e^n}{n^2} \sim \text{"}\frac{\infty}{\infty}\text{"}$

$$\overset{H}{=} \lim_{n\to\infty} \frac{e^n}{2n} \sim \text{"}\frac{\infty}{\infty}\text{"}$$

$$\overset{H}{=} \lim_{n\to\infty} \frac{e^n}{2} = \infty$$

Since $\displaystyle\lim_{n\to\infty} \frac{e^n}{n^2} \neq 0$,

series $\boxed{\text{diverges}}$

[3]