Question 1: (Integration by Parts) Determine $\int \frac{\ln(x)}{\sqrt{x}}\,dx = I$

$u = \ln(x) \quad dv = \frac{1}{\sqrt{x}}\,dx$

$du = \frac{1}{x}\,dx \quad v = 2\sqrt{x}$

$$I = \int u\,dv = uv - \int v\,du$$

$$= \ln(x)\cdot 2\sqrt{x} - \int 2\sqrt{x}\,\frac{1}{x}\,dx$$

$$= 2\sqrt{x}\ln(x) - 2\int \frac{1}{\sqrt{x}}\,dx$$

$$= \boxed{2\sqrt{x}\ln(x) - 4\sqrt{x} + C}$$

[5]

Question 2: (Integration by Parts) Determine $\int_0^1 \cos^{-1}(x)\,dx$

For $I = \int \cos^{-1}(x)\,dx$:

Let $u = \cos^{-1}(x) \quad dv = 1\cdot dx$

$du = \frac{-1}{\sqrt{1-x^2}}\,dx \quad v = x$

$$I = \int u\,dv = uv - \int v\,du$$

$$= x\cos^{-1}(x) - \int x\cdot\frac{-1}{\sqrt{1-x^2}}\,dx$$

$$= x\cos^{-1}(x) - \left(\frac{1}{2}\right)\int \frac{-2x}{\underbrace{\sqrt{1-x^2}}_{u = 1-x^2,\ du = -2x\,dx}}\,dx$$

$$= x\cos^{-1}(x) - \sqrt{1-x^2} + C$$

So $\int_0^1 \cos^{-1}(x)\,dx$

$$= \left[x\cos^{-1}(x) - \sqrt{1-x^2}\right]_0^1$$

$$= \left(1\cdot\cancel{\cos^{-1}(1)}^{\,0} - \cancel{\sqrt{1-1^2}}^{\,0}\right) - \left(\cancel{0\cdot\cos^{-1}(0)}^{\,0} - \sqrt{1-0^2}\right)$$

$$= \boxed{1}$$

[5]

Question 3: (Trigonometric Substitution) Determine $\int \frac{1}{x^2\sqrt{16-x^2}}\,dx = I$

Let $x = 4\sin\theta$

$dx = 4\cos\theta\, d\theta$

$$I = \int \frac{1}{16\sin^2\theta\sqrt{16-16\sin^2\theta}} \cdot 4\cos\theta\, d\theta$$

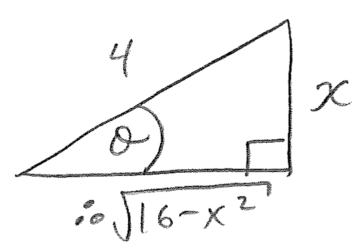
$$= \frac{4}{16}\int \frac{\cos\theta}{\sin^2\theta\cdot\sqrt{16(1-\sin^2\theta)}}\, d\theta$$

$$= \frac{\cancel{4}}{16}\int \frac{\cancel{\cos\theta}}{\sin^2\theta\cdot\cancel{4}\cdot\cancel{\cos\theta}}\, d\theta$$

$$= \frac{1}{16}\int \csc^2\theta\, d\theta$$

$$= \frac{-1}{16}\cot\theta + C$$

$\sin\theta = \frac{x}{4}$:



$$= \boxed{\frac{-1}{16}\,\frac{\sqrt{16-x^2}}{x} + C}$$

[10]

Question 4: (Partial Fractions) Determine $\int \frac{x-4}{x^2-5x+6}\,dx = I$

$$\frac{x-4}{x^2-5x+6} = \frac{x-4}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3} = \frac{(A+B)x + (-3A-2B)}{(x-2)(x-3)}$$

$$\therefore \left.\begin{aligned} A+B &= 1 \\ -3A-2B &= -4 \end{aligned}\right\} \begin{aligned} &\Rightarrow B = 1-A \\ &\Rightarrow -3A-2(1-A) = -4 \Rightarrow -A = -2 \Rightarrow A = 2, \therefore B = 1-2=-1 \end{aligned}$$

$$\therefore \quad I = \int \frac{2}{x-2} + \frac{-1}{x-3}\,dx$$

$$= \boxed{2\ln|x-2| - \ln|x-3| + C}$$

[5]

Question 5: Determine $\int \frac{4x^2+4x-1}{4x^2-4x+3}\,dx = I$

$$\begin{array}{r|l} & \quad 1 \\ \hline 4x^2-4x+3 & 4x^2+4x-1 \\ & -(4x^2-4x+3) \\ \hline & \quad 8x-4 \end{array}$$

$$\therefore \quad I = \int 1 + \frac{8x-4}{4x^2-4x+3}\,dx$$

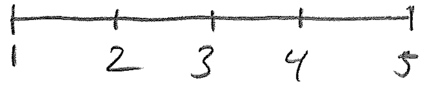
$$= \int 1\,dx + \underbrace{\int \frac{8x-4}{4x^2-4x+3}\,dx}_{\substack{u = 4x^2-4x+3 \\ du = (8x-4)dx}}$$

$$\boxed{\therefore I = x + \ln|4x^2-4x+3| + C}$$

[5]

Question 6: Use T_4, the Trapezoid Rule on four subintervals to approximate $\int_1^5 \frac{\cos(\pi x)}{x}\,dx$. Express your final answer as a single simplified fraction.

$\Delta x = \frac{b-a}{n} = \frac{5-1}{4} = 1$ $\therefore$



$f(x) = \frac{\cos(\pi x)}{x}$

$$\therefore\ I \approx T_4 = \frac{\Delta x}{2}\left[f(1) + 2f(2) + 2f(3) + 2f(4) + f(5)\right]$$

$$= \frac{1}{2}\left[\frac{\cos(\pi)}{1} + \frac{\cancel{2}\cos(2\pi)}{\cancel{2}} + 2\frac{\cos(3\pi)}{3} + 2\frac{\cos(4\pi)}{4} + \frac{\cos(5\pi)}{5}\right]$$

$$= \frac{1}{2}\left[\cancel{-1} + \cancel{1} - \frac{2}{3} + \frac{1}{2} - \frac{1}{5}\right]$$

$$= \frac{1}{2}\left[\frac{-20+15-6}{30}\right] = \boxed{-\frac{11}{60}}$$

[5]

Question 7: Determine whether $\int_0^\infty x^2 e^{-x^3}\,dx$ converges or diverges. If it converges give the value, if it diverges then say so. Make proper use of any required limits and use proper notation.

For $I = \int x^2 e^{-x^3}\,dx$, let $u = -x^3$, $du = -3x^2\,dx$

$$\therefore\ I = -\frac{1}{3}\int e^u\,du = -\frac{1}{3}e^u + C = -\frac{1}{3}e^{-x^3} + C.$$

$$\int_0^\infty x^2 e^{-x^3}\,dx = \lim_{b\to\infty}\int_0^b x^2 e^{-x^3}\,dx$$

$$= \lim_{b\to\infty}\left[-\frac{1}{3}e^{-x^3}\right]_0^b$$

$$= \lim_{b\to\infty}\left[\underbrace{-\frac{1}{3}e^{-b^3}}_{\to 0} - \left(-\frac{1}{3}\overset{1}{e^{0}}\right)\right]$$

$= \frac{1}{3}$ $\therefore$ integral converges to $\boxed{\frac{1}{3}}$

[5]

Question 8: Determine if the improper integral $\int_0^5 \frac{x}{x-2}\,dx$ converges or diverges. If it converges give the value, if it diverges then say so. Make proper use of any required limits and use proper notation.

Note the integrand $\frac{x}{x-2}$ has a discontinuity at $x=2$.

$$\text{So } I = \int_0^5 \frac{x}{x-2}\,dx = \underbrace{\int_0^2 \frac{x}{x-2}\,dx}_{I_1} + \underbrace{\int_2^5 \frac{x}{x-2}\,dx}_{I_2}.$$

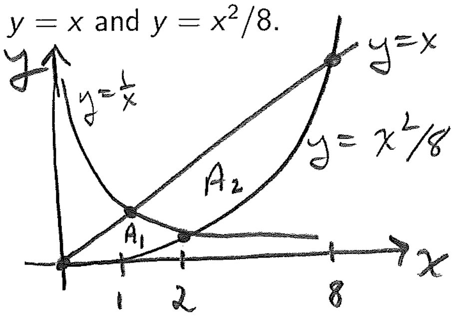
For I_1: $x-2 \overline{)\,x\,}$ with quotient 1; $-(x-2)$; remainder 2

$$\therefore I_1 = \int_0^2 \left(1+\frac{2}{x-2}\right)dx$$
$$= \lim_{b\to 2^-} \int_0^b \left(1+\frac{2}{x-2}\right)dx$$
$$= \lim_{b\to 2^-} \Big[x + 2\ln|x-2|\Big]_0^b$$
$$= \lim_{b\to 2^-} \Big[\big(b+\underbrace{2\ln|b-2|}_{\to -\infty}\big) - \big(0+2\ln|0-2|\big)\Big]$$
$$= -\infty$$

$\boxed{\therefore \text{ Since } I_1 \text{ diverges, } I \text{ diverges also.}}$

[5]

Question 9: Determine the area of the region in the first quadrant that is bounded by the curves $y = 1/x$, $y = x$ and $y = x^2/8$.



This problem was unintentionally ambiguous. The bounded region could be A_1, A_2, or A_1 together with A_2.

$$A_1 = \int_0^1 \left(x-\frac{x^2}{8}\right)dx + \int_1^2 \left(\frac{1}{x}-\frac{x^2}{8}\right)dx = \boxed{\ln(2)+\frac{1}{6}}$$

$$A_2 = \int_1^2 \left(x-\frac{1}{x}\right)dx + \int_2^8 \left(x-\frac{x^2}{8}\right)dx = \boxed{\frac{21}{2}-\ln(2)}$$

$$A_1 + A_2 = \int_0^8 \left(x-\frac{x^2}{2}\right)dx = \boxed{\frac{32}{3}}$$

If you gave any of these three answers then you get full marks. (The intended region was A_1).

[5]
