Question 1: For this question use the graph of $y = g(x)$ below. Note that each section of the graph is either a straight line or an arc of a circle of radius 1.

$A_1 = \frac{1}{2}$

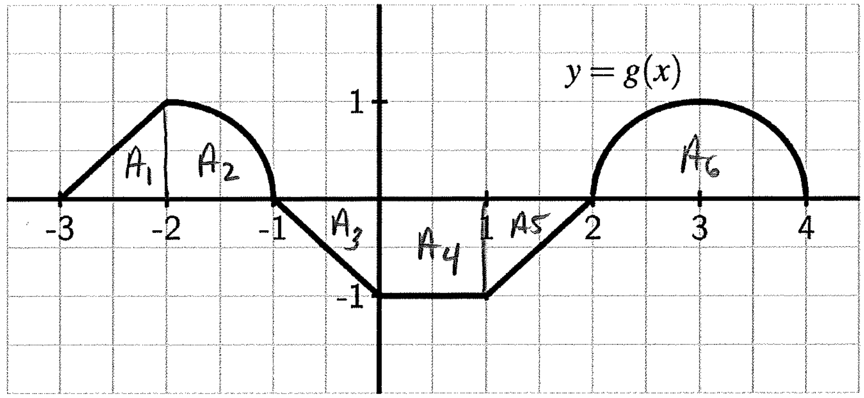
$A_2 = \frac{\pi}{4}$

$A_3 = \frac{1}{2}$

$A_4 = 1$

$A_5 = \frac{1}{2}$

$A_6 = \frac{\pi}{2}$



(a) Determine $\int_{-3}^{0} g(x)\,dx = A_1 + A_2 - A_3 = \boxed{\frac{\pi}{4}}$

[3]

(b) Determine the average value of g over the interval $[-3, 4]$.

$$g_{ave} = \frac{1}{4-(-3)} \int_{-3}^{4} g(x)\,dx$$

$$= \frac{1}{7}\left[A_1 + A_2 - A_3 - A_4 - A_5 + A_6\right]$$

$$= \frac{1}{7}\left[\cancel{\tfrac{1}{2}} + \frac{\pi}{4} - \cancel{\tfrac{1}{2}} - 1 - \frac{1}{2} + \frac{\pi}{2}\right] = \frac{1}{7}\left(\frac{3\pi}{4} - \frac{3}{2}\right)$$

$$= \boxed{\frac{3\pi - 6}{28}}$$

[4]

(c) Let $G(u) = \int_{-3}^{u^2} g(x)\,dx$. Find $G'(1)$.

$$G'(u) = g(u^2) \cdot 2u$$

Chain Rule.

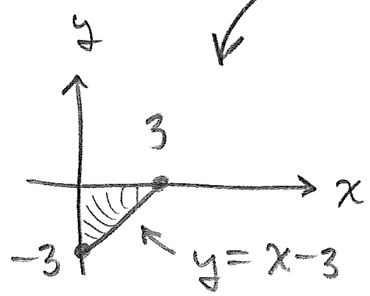
$$G'(1) = g(1^2) \cdot (2)(1)$$

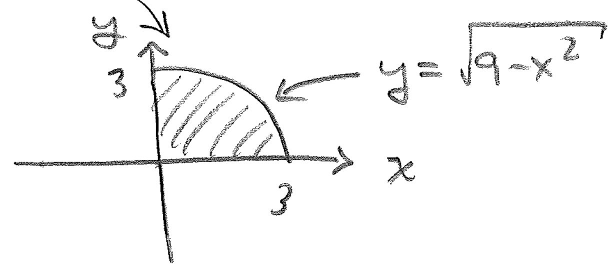
$$= (-1)(2) = \boxed{-2}$$

[3]

Question 2: Evaluate $\int_0^3 x - 3 + \sqrt{9 - x^2}\, dx$ (Hint: area interpretation.)

$$I = \int_0^3 (x-3)\, dx + \int_0^3 \sqrt{9-x^2}\, dx$$





$$\therefore\ I = -\frac{1}{2}(3^2) + \frac{1}{4}\pi \cdot 3^2$$
$$= \boxed{\frac{9\pi - 18}{4}}$$

[5]

Question 3: A garden centre sells cedar trees which increase in height at a rate of $r(t) = (1 + \sqrt{t})/2$ metres per year once the customer takes them home and transplants them in the ground. If a newly transplanted tree is 1 metre tall, determine the height of the tree four years later.

H = initial height + change in height

$$= 1 + \int_0^4 \frac{1+\sqrt{t}}{2}\, dt$$
$$= 1 + \int_0^4 \left(\frac{1}{2} + \frac{1}{2} t^{\frac{1}{2}}\right) dt$$
$$= 1 + \frac{1}{2}[t]_0^4 + \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3}\left[t^{3/2}\right]_0^4$$
$$= 1 + \frac{1}{2}(4-0) + \frac{1}{3}(4^{3/2} - 0)$$
$$= 1 + 2 + \frac{8}{3} = \boxed{\frac{17}{3}\ \text{m}}$$

[5]

Question 4: Determine $f(4)$ if
$$\int_0^x f(t)\,dt = x\cos(\pi x)$$

$$\frac{d}{dx}\left[\int_0^x f(t)\,dt\right] = \frac{d}{dx}\left[x\cos(\pi x)\right]$$

$$f(x) = \cos(\pi x) - \pi x\sin(\pi x)$$

$$f(4) = \cos(4\pi) - 4\pi\cancel{\sin(4\pi)}^{\,0}$$

$$= \boxed{1}$$

[5]

Question 5: The outside temperature in degrees Celsius is given by the function $T(t) = 10 + 3\sin(\pi t/12)$ where t is in hours and $t = 0$ corresponds to 6:00 AM. Determine the average temperature over the period from 6:00 AM to 6:00 PM of the same day.

$$T_{ave} = \frac{1}{12-0}\int_0^{12} 10 + 3\sin\left(\frac{\pi}{12}t\right)\,dt$$

$$= \frac{1}{12}\left[10t - 3\frac{\cos\left(\frac{\pi}{12}t\right)}{\frac{\pi}{12}}\right]_0^{12}$$

$$= \frac{1}{12}\left[10t - \frac{36}{\pi}\cos\left(\frac{\pi}{12}t\right)\right]_0^{12}$$

$$= \frac{1}{12}\left[\left((10)(12) - \frac{36}{\pi}\underbrace{\cos\left(\cancel{\tfrac{\pi}{12}}\cdot\cancel{12}\right)}_{-1}\right) - \left(\cancel{(10)(0)}^{\,0} - \left(\frac{36}{\pi}\right)\underbrace{\cos(0)}_{1}\right)\right]$$

$$= \frac{1}{12}\left[120 + \frac{36}{\pi} + \frac{36}{\pi}\right] = \boxed{10 + \frac{6}{\pi}\text{ degrees}}$$

[5]

Question 6: Evaluate the following definite integrals:

(a) $\int_{-3}^{4}\left(5-\frac{x}{2}\right)dx$ $= \left[5x - \frac{x^2}{4}\right]_{-3}^{4}$

$= \left((5)(4) - \frac{(4)^2}{4}\right) - \left((5)(-3) - \frac{(-3)^2}{4}\right)$

$= 20 - 4 + 15 + \frac{9}{4}$

$= \boxed{\frac{133}{4}}$

[2]

(b) $\int_{0}^{\pi}\left(\frac{\sin(x)+\pi\cos(x)}{2}\right)dx$ $= \frac{1}{2}\left[-\cos(x)+\pi\sin(x)\right]_0^{\pi}$

$= \frac{1}{2}\left[\left(-\cos(\pi)+\cancel{\pi\sin(\pi)}^{\,0}\right)-\left(-\cos(0)+\cancel{\pi\sin(0)}^{\,0}\right)\right]$

$= \frac{1}{2}(1+1) = \boxed{1}$

[2]

(c) $\int_{-1}^{2}(t+1)(t^2+4)\,dt$ $= \int_{-1}^{2}(t^3+t^2+4t+4)\,dt$

$= \frac{1}{4}\left[t^4\right]_{-1}^{2} + \frac{1}{3}\left[t^3\right]_{-1}^{2} + \frac{4}{2}\left[t^2\right]_{-1}^{2} + 4\left[t\right]_{-1}^{2}$

$= \frac{1}{4}[16-1] + \frac{1}{3}[8+1] + 2[4-1] + 4[2+1]$

$= \frac{15}{4} + 3 + 6 + 12 = \boxed{\frac{99}{4}}$

[3]

(d) $\int_{1}^{e}\left(e^{x-1}-\frac{3}{x}\right)dx$

$= \int_1^e\left(e^{-1}\cdot e^{x} - \frac{3}{x}\right)dx$

$= e^{-1}\left[e^x\right]_1^e - 3\left[\ln|x|\right]_1^e$

$= \frac{e^e - e}{e} - 3\left[\cancel{\ln|e|}^{\,1} - \cancel{\ln|1|}^{\,0}\right]$

$= e^{e-1} - 1 - 3$

$= \boxed{e^{e-1} - 4}$

[3]

Question 7: (Substitution Method) Determine the following:

(a) $\int_1^2 \frac{\cos(1-1/x)}{x^2}\,dx$

$u = 1 - \frac{1}{x}$ $\quad x=1 \Rightarrow u=0$

$du = \frac{1}{x^2}$ $\quad x=2 \Rightarrow u=\frac{1}{2}$

$I = \int_0^{\frac{1}{2}} \cos(u)\,du$

$= [\sin(u)]_0^{1/2} = \boxed{\sin(\tfrac{1}{2})}$

[2]

(b) $\int \sec^2(3x+2)\,dx = \boxed{\frac{\tan(3x+2)}{3} + C}$ $\leftarrow$ $u = 3x+2$, $du = 3dx$

[2]

(c) $\frac{1}{3}\int 3e^x\sqrt{1+3e^x}\,dx$

$u = 1+3e^x$

$du = 3e^x dx$

$= \frac{1}{3}\int \sqrt{u}\,du$

$= \frac{1}{3}\,\frac{u^{3/2}}{3/2} + C$

$= \boxed{\frac{2}{9}(1+3e^x)^{3/2} + C}$

[2]

(d) $\int_e^{e^4} \frac{1}{x\sqrt{\ln(x)}}\,dx$

$u = \ln(x)$ $\quad x=e \Rightarrow u=1$

$du = \frac{1}{x}dx$ $\quad x=e^4 \Rightarrow u=4$

$I = \int_1^4 \frac{1}{\sqrt{u}}\,du = 2\left[\sqrt{u}\right]_1^4$

$= 2(\sqrt{4}-\sqrt{1})$

$= \boxed{2}$

[4]