MATH 122: Calculus II

Section 8.7 Taylor and Maclaurin Series

8.7 Taylor and Maclaurin Series

In 8.6 we obtained power series representations for certain special functions.

In 8.7 we look at the more general problem, namely, which functions have power series representations, and how can we find them?

8.7 Taylor and Maclaurin Series

Suppose that the function f can be represented by a power series:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$$

for $|x-a| < R$.

Then for $|x-a| < R$

$$\begin{aligned} f'(x) &= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + 5c_5(x-a)^4 + \cdots \\ f''(x) &= 2c_2 + 2\cdot 3c_3(x-a) + 3\cdot 4c_4(x-a)^2 + 4\cdot 5c_5(x-a)^3 + \cdots \\ f^{(3)}(x) &= 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4(x-a) + 3\cdot 4\cdot 5c_5(x-a)^2 + \cdots \\ f^{(4)}(x) &= 2\cdot 3\cdot 4c_4 + 2\cdot 3\cdot 4\cdot 5c_5(x-a) + \cdots \end{aligned}$$

This leads to

$$f(a) = c_0$$

$$f'(a) = c_1$$

$$f''(a) = 2c_2 = 2!c_2$$

$$f^{(3)}(a) = 2\cdot 3c_3 = 3!c_3$$

$$f^{(4)}(a) = 2\cdot 3\cdot 4c_4 = 4!c_4$$

8.7 Taylor and Maclaurin Series

In general: $f^{(n)}(a) = 2 \cdot 3 \cdot 4 \cdots nc_n = n!c_n$. Solving for c_n we obtain

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Theorem 5

If f has a power series representation at a, that is if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

8.7 Taylor and Maclaurin Series

If f has a power series representation at a, it must be of the following form:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

This is called the **Taylor series of** f **at** a (or **about** a or **centered at** a).

8.7 Taylor and Maclaurin Series

For the special case $a = 0$ we have that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots$$

This called the **Maclaurin series** of f.

8.7 Taylor and Maclaurin Series

Example 1

Find the Maclaurin series of $f(x) = \cos(x)$ and its radius of convergence.

The Maclaurin series for $f(x)$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

$$\begin{aligned} f(x) &= \cos(x) & f(0) &= 1 \\ f'(x) &= -\sin(x) & f'(0) &= 0 \\ f''(x) &= -\cos(x) & f''(0) &= -1 \\ f^{(3)}(x) &= \sin(x) & f^{(3)}(0) &= 0 \\ f^{(4)}(x) &= \cos(x) & f^{(4)}(0) &= 1 \end{aligned}$$

Example 1 (continued)

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{1!}x^4 + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

To determine the radius of convergence, we will use the Ratio Test. We have that

$$a_n = (-1)^n \frac{x^{2n}}{(2n)!} \quad \text{and} \quad a_{n+1} = (-1)^{n+1} \frac{x^{2n+2}}{(2n+2)!}$$

Example 1 (continued)

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}x^{2n+2}}{(2n+2)!} \times \frac{(2n)!}{(-1)^n x^{2n}}\right| = \frac{x^2}{(2n+2)(2n+1)}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{x^2}{(2n+2)(2n+1)} = 0$$

Notice that the limit above is always zero regardless of the value of x.

That is, we always obtain a limit of zero, which according to the Ratio Test (since this limit of zero is less than one), says that the radius of convergence is $R = \infty$. That is, the interval of convergence is $(-\infty, \infty)$.

8.7 Taylor and Maclaurin Series

Example 1 says that **if** $\cos(x)$ has a power series representation at $a = 0$, then

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

The fact that the series on the right converges for all values of x, does not mean that it converges to $\cos(x)$, necessarily.

There are functions that are not equal to their Taylor series.

8.7 Taylor and Maclaurin Series

Think of the Taylor series as something that has its own existence. Sometimes the function and its Taylor series will equal each other, but there's no guarantee that it will always happen.

How can we determine whether a function does have a power series representation?

To show that a function is actually equal to its Taylor series we have to examine the partial sums in the Taylor series and show that they converge to the function we have in mind.

The partial sums in the case of Taylor series are

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$T_n(a) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

This is called the ***n*-th degree Taylor polynomial of f at a**.

8.7 Taylor and Maclaurin Series

Let $f(x)$ be the function we are trying to represent by its Taylor series.

Define the **remainder** of the Taylor series by $R_n(x) = f(x) - T_n(x)$.

This means that

$$f(x) = T_n(x) + R_n(x).$$

If we can show that $\lim_{n\to\infty} R_n(x) \to 0$, it will then follow that

$$f(x) = \lim_{n\to\infty} T_n(x),$$

meaning that the partial sums will converge to $f(x)$.

8.7 Taylor and Maclaurin Series

To show that $R_n(x) \to 0$ as $n \to \infty$, we often use the following result, called Taylor's Formula.

Taylor's Formula

If f has $n+1$ derivatives in an interval I that contains the number a, then for x in I there is a number z strictly between x and a such that the remainder in the Taylor series can be expressed as

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}.$$

The remainder term is very similar to the terms in the Taylor series. The only difference is that $f^{(n+1)}$ is evaluated at z instead of a.

In the case of $f(x) = \cos(x)$, $f^{(n+1)}(z) = \pm\cos(z)$ or $\pm\sin(z)$. This would mean that $|f^{(n+1)}(z)| \leq 1$. For the Maclaurin series of $\cos(x)$ where $a = 0$,

$$0 \leq |R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!}(x-0)^{n+1} \right| \leq \frac{|x^{n+1}|}{(n+1)!}.$$

As $n \to \infty$, $\dfrac{|x^{n+1}|}{(n+1)!} \to 0$ for all x.

Therefore, $|R_n(x)| \to 0$, so that $R_n(x) \to 0$ as $n \to \infty$.

This means that $\cos(x)$ does equal its Taylor series:

$$\begin{aligned} \cos(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \end{aligned} \qquad \text{for every real number } x$$

Example 2

Find the Maclaurin series for $\sin(x)$.

In this case we could compute all of the derivatives of $\sin(x)$ and evaluate them at $a = 0$ to build the Maclaurin series from scratch, but there is an easier way.

We start with the series for $\cos(x)$ instead

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

and differentiate both sides. Here we will be using Theorem 2 from 8.6 that says that we can differentiate the power series as if it is a polynomial, and the interval of convergence will remain the same.

Example 2 (continued)

Recall that $\frac{d}{dx}[\cos(x)] = -\sin(x)$, or that $-\frac{d}{dx}[\cos(x)] = \sin(x)$.

$$\begin{aligned}
\sin(x) &= -\frac{d}{dx}\left[\cos(x)\right] \\
&= -\frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\right] \\
&= -\sum_{n=0}^{\infty}\frac{d}{dx}\left[(-1)^n\frac{x^{2n}}{(2n)!}\right] \\
&= -\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n-1}}{(2n-1)!} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!}
\end{aligned}$$

Example 2 (continued)

Notice that

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty}(-1)^{n}\frac{x^{2n+1}}{(2n+1)!}$$

Since the Maclaurin series for $\cos(x)$ converges for all x, Theorem 2 in 8.6, says that the differentiated series for $\sin(x)$ also converges for all x. Therefore,

$$\begin{aligned}\sin(x) &= \sum_{n=0}^{\infty}(-1)^{n}\frac{x^{2n+1}}{(2n+1)}! \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\end{aligned} \qquad \text{for all } x$$

8.7 Taylor and Maclaurin Series

The power series we obtained for $\sin(x)$ in Example 2 (in an indirect manner) will in fact equal the Taylor series (Maclaurin series) for $\sin(x)$ that can be obtained using direct methods.

The reason for this is Theorem 5 in this section. It states that no matter how a power series representation, $f(x) = \sum c_n(x-a)^n$, is obtained, it is always the case that

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

That is, the coefficients are unique.

In Example 1 on page 477 of the textbook it shown that the Maclaurin series for $f(x) = e^x$ is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

It is also shown that the radius of convergence of this series is $R = \infty$.

In Example 2 on page 479 it is shown that e^x is actually equal to its Maclaurin series. This is done by showing that the remainder $R_n(x)$ goes to zero as $n \to \infty$. This means that

$$\boxed{\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \end{aligned} \qquad \text{for all } x}$$

Setting $x = 1$ in the series above we obtain

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Example 3

Using the definition of Taylor series, find the Taylor series for $f(x) = \cos(x)$ centered at $a = \pi/2$.

The definition of the Taylor series for $f(x)$ at $a = \pi/2$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/2)}{n!}(x - \pi/2)^n.$$

$$f(x) = \cos(x) \qquad f(\pi/2) = 0$$

$$f'(x) = -\sin(x) \qquad f'(\pi/2) = -1$$

$$f''(x) = -\cos(x) \qquad f''(\pi/2) = 0$$

$$f^{(3)}(x) = \sin(x) \qquad f^{(3)}(\pi/2) = 1$$

$$f^{(4)}(x) = \cos(x) \qquad f^{(4)}(\pi/2) = 0$$

Example 3 (continued)

$$f(\pi/2) + \frac{f'(\pi/2)}{1!}(x - \pi/2) + \frac{f''(\pi/2)}{2!}(x - \pi/2)^2 +$$

$$\frac{f^{(3)}(\pi/2)}{3!}(x - \pi/2)^3 + \frac{f^{(4)}(\pi/2)}{1!}(x - \pi/2)^4 + \cdots$$

$$= -(x - \pi/2) + \frac{1}{3!}(x - \pi/2)^3 - \frac{1}{5!}(x - \pi/2)^5 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}(x - \pi/2)^{2n+1}$$

The Maclaurin series ($a = 0$) for $\cos(x)$ in Example 1 can be used to approximate $\cos(x)$ for values of x close to $a = 0$.

The Taylor series for $\cos(x)$ centered at $a = \pi/2$ in Example 3 can be used to approximate $\cos(x)$ for values of x close to $a = \pi/2$.

We will say more about this in the next section.

Example 4

Use the Maclaurin series for $\cos(x)$ to obtain the Maclaurin series for $x\cos(2x)$.

The Maclaurin series for $\cos(x)$ is

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Substituting $2x$ for x above gives

$$\cos(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} x^{2n}$$

The indirect method of obtaining the power series for $\cos(2x)$ produces the same series as the direct method using the definition of Taylor series. Theorem 5 guarantees that the coefficients are unique.

Example 4 (continued)

Multiplying this series by x gives the Maclaurin series for $x\cos(2x)$:

$$x\cos(2x) = x\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}}{(2n)!}x^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}}{(2n)!}x^{2n+1}$$

Theorem 5, again, guarantees that this will be the series for $x\cos(x)$.

This series converges for all x since the original series converges for all x.

Listed on the next slide are some important Maclaurin series together with the radius of convergence of each one.

The series for $\ln(1 + x)$ is derived in Example 6 on page 472.

8.7 Taylor and Maclaurin Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

Example 5

Evaluate $\int e^{-x^2}\,dx$ as an infinite series.

Recall that e^{-x^2} cannot be integrated by any of our techniques since it does not have an elementary antiderivative. This example illustrates one of the main uses of Taylor series: express e^{-x^2} as a series and integrate it term-by-term.

We obtain a series representation for e^{-x^2} by substituting $-x^2$ for x in the Maclaurin series for e^x:

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$

Example 5 (continued)

Integrating term-by-term we get

$$
\begin{aligned}
\int e^{-x^2}\,dx &= \int \left(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots \right) dx \\
&= C + x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \cdots \\
&= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)n!}
\end{aligned}
$$

This series converges for all x since the original series converges for all x.

Example 6

Use series to evaluate the limit $\lim_{x\to 0} \dfrac{1-\cos(x)}{1+x-e^x}$.

$$\lim_{x\to 0} \frac{1-\cos(x)}{1+x-e^x} = \lim_{x\to 0} \frac{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)}{1+x-\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)}$$

$$= \lim_{x\to 0} \frac{\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\cdots}{-\frac{x^2}{2!}-\frac{x^3}{3!}-\frac{x^4}{4!}-\cdots}$$

$$= \lim_{x\to 0} \frac{\frac{1}{2!}-\frac{x^2}{4!}+\frac{x^4}{6!}-\cdots}{-\frac{1}{2!}-\frac{x}{3!}-\frac{x^2}{4!}-\cdots} = \frac{\frac{1}{2}-0}{-\frac{1}{2}-0} = -1$$

8.7 Taylor and Maclaurin Series

Convergent power series behave like polynomials: they can be added, subtracted, multiplied and divided just like we would do with ordinary polynomials.

Example 7

Find the first four nonzero terms in the Maclaurin series for $\dfrac{e^x}{1-x}$.

Since $\dfrac{e^x}{1-x} = e^x\left(\dfrac{1}{1-x}\right)$, we multiply the two corresponding Maclaurin series.

$$e^x\left(\frac{1}{1-x}\right) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)(1 + x + x^2 + x^3 + \cdots)$$

Example 7

$$e^x\left(\frac{1}{1-x}\right) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 + x + x^2 + x^3 + \cdots\right)$$

$$= 1\cdot\left(1 + x + x^2 + x^3 + \cdots\right) + x\cdot\left(1 + x + x^2 + x^3 + \cdots\right)$$

$$+\frac{x^2}{2}\cdot\left(1 + x + x^2 + x^3 + \cdots\right) + \frac{x^3}{6}\cdot\left(1 + x + x^2 + x^3 + \cdots\right) + \cdots$$

$$= 1 + 2x + \left(x^2 + x^2 + \frac{x^2}{2}\right) + \left(x^3 + x^3 + \frac{x^3}{2} + \frac{x^3}{6}\right) + \cdots$$

$$= 1 + 2x + \frac{5}{2}x^2 + \frac{16}{6}x^3 + \cdots$$

For an example involving the division of two Maclaurin series, see Example 11 on page 485.

It uses a process similar to long division.