MATH 122: Calculus II

Section 8.6 Representing Functions as Power Series

8.6 Representing Functions as Power Series

This section deals with representing certain functions as power series either by starting with a geometric series and manipulating it or by differentiating or integrating a known series.

The reason for doing this is that it provides us with a way of integrating functions that don't have elementary antiderivatives, for solving certain differential equations, or for approximating functions by polynomials.

8.6 Representing Functions as Power Series

Consider the series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

It is a geometric series with $a = 1$ and $r = x$. We know that it will converge if $|r| = |x| < 1$. In fact, it will converge to

$$\frac{a}{1-r} = \frac{1}{1-x}.$$

Therefore

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \quad \text{provided that } |x| < 1.$$

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It is important to realize what we mean when we say that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \quad \text{provided that } |x| < 1.$$

If $|x| < 1$, then the function on the left and the power series on the right produce the exact same result.

If $|x| \geq 1$, we may still be able to compute with the function on the left (except when $x = 1$), but the power series diverges. For these values of x, the two sides of the equation give different results.

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Example 1

Find a power series representation of $f(x) = \dfrac{1}{1-x^3}$ and state the interval of convergence.

We know that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \quad \text{provided that } |x| < 1.$$

Replace x by x^3 in the equation above:

$$\begin{aligned} \frac{1}{1-x^3} &= 1 + x^3 + (x^3)^2 + (x^3)^3 + \cdots \\ &= 1 + x^3 + x^6 + x^9 + \cdots = \sum_{n=0}^{\infty} x^{3n} \end{aligned}$$

Example 1 (continued)

$$\frac{1}{1-x^3} = 1 + x^3 + x^6 + x^9 + \cdots = \sum_{n=0}^{\infty} x^{3n}$$

The right-hand side is a geometric series with $r = x^3$. It will converge when $|x^3| < 1$, which is that same as $-1 < x^3 < 1$, meaning that $-1 < x < 1$.

The radius of convergence is $R = 1$, and the interval of convergence is $(-1, 1)$.

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Example 2

Find a power series representation for $g(x) = \dfrac{1}{x+3}$ and state the interval of convergence.

We want to use the series for $\dfrac{1}{1-x}$ again.

Before we can use it, we need to make $g(x)$ "look more like" $\dfrac{1}{1-x}$.

$$\frac{1}{x+3} = \frac{1}{3\left(1+\frac{x}{3}\right)} = \frac{1}{3\left[1-\left(-\frac{x}{3}\right)\right]}$$

This shows that to find the series for $g(x)$ we multiply the series for $1/(1-x)$ by $1/3$ and we substitute $(-x/3)$ for x in the series for $1/(1-x)$.

Example 2 (continued)

Therefore

$$g(x) = \frac{1}{3\left[1 - \left(-\frac{x}{3}\right)\right]} = \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}x^n.$$

This series converges when $|-x/3| < 1$, which is the same as $|x| < 3$.

The radius of convergence is $R = 3$ and the interval of convergence is $(-3, 3)$.

Example 2 (continued)

Using the power series representation for $g(x) = \dfrac{1}{x+3}$, find a power series representation for $h(x) = \dfrac{x}{x+3}$.

We multiply the series just obtained by x to produce the series for $h(x)$:

$$h(x) = \frac{x}{x+3} = x\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}}x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}}x^{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^n}x^n.$$

The interval of convergence is the same as before: $(-3, 3)$.

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Theorem 2: Differentiation and Integration of Power Series

If the power series $\sum c_n(x-a)^n$ has a radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on $(a-R, a+R)$.

Moreover

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(ii) $\int f(x)\,dx = C + c_0(x-a) + c_1\dfrac{(x-a)^2}{2} + c_2\dfrac{(x-a)^3}{3} + \cdots =$
$C + \sum_{n=0}^{\infty} c_n \dfrac{(x-a)^{n+1}}{n+1}$

The radii of convergence of the power series in (i) and (ii) are both R.

Notice that in (ii) above we did the following:

$$\int c_0\,dx = c_0 x + C_1 = c_0(x-a) + C,$$

where $C = C_1 + c_0 a$. This is done so that all of the terms in (ii) have the same form.

Theorem 2 says that we can differentiate and integrate a power series by differentiating and integrating each term individually. In this regard, power series behave like polynomials.

Note 1

Equations (i) and (ii) can be rewritten as

- $$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}[c_n(x-a)^n]$$

- $$\int \left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] dx = \sum_{n=0}^{\infty} \int c_n(x-a)^n dx$$

These are true provided we are dealing with *power series*. They may not hold for other types of series.

Note 2

Theorem 2 states that the radius of convergence stays the same when a power series is differentiated or integrated. This does not mean that the *interval* of convergence remains the same.

For instance the original series for $f(x)$ might converge at an endpoint, but the differentiated series may fail to converge at this same endpoint.

Example 3

Find a power series representation for $f(x) = \dfrac{x^3}{(x-2)^2}$ and state the interval of convergence.

We focus first on finding a power series representation for $\dfrac{1}{(x-2)^2}$. Once we have that, we will multiply it by x^3.

Notice that

$$\frac{d}{dx}\left[\frac{1}{x-2}\right] = \frac{-1}{(x-2)^2} \quad \text{or} \quad \frac{d}{dx}\left[\frac{-1}{x-2}\right] = \frac{1}{(x-2)^2}.$$

This says that we should find a power series representation for $\dfrac{-1}{x-2}$, and then differentiate it term-by-term to obtain a series for $\dfrac{1}{(x-2)^2}$.

Example 3 (continued)

We want to use the series for $1/(1-x)$ to obtain a series for $-1/(x-2) = 1/(2-x)$. Notice that

$$\frac{1}{2-x} = \frac{1}{2\left(1-\frac{x}{2}\right)}.$$

Therefore, we replace x by $x/2$ in the series for $1/(1-x)$ and also multiply it by $1/2$:

$$\frac{1}{2-x} = \frac{1}{2\left(1-\frac{x}{2}\right)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}}.$$

This series converges when $|x/2| < 1$, that is $|x| < 2$, or $-2 < x < 2$.

Example 3 (continued)

$$\frac{1}{2-x} = \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}$$

Compute the derivative on both sides:

$$\frac{d}{dx}\left[\frac{1}{2-x}\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}\right].$$

This leads to:

$$\frac{1}{(x-2)^2} = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{2^{n+1}}.$$

This series converges on the same interval as the series for $1/(2-x)$, namely, $(-2, 2)$.

Example 3 (continued)

To obtain the series representation for $\dfrac{x^3}{(x-2)^2}$, multiply the last series by x^3.

$$\frac{x^3}{(x-2)^2} = x^3 \frac{1}{(x-2)^2} = x^3 \sum_{n=1}^{\infty} \frac{nx^{n-1}}{2^{n+1}} = \sum_{n=1}^{\infty} \frac{nx^{n+2}}{2^{n+1}} = \sum_{n=3}^{\infty} \frac{(n-2)x^n}{2^{n-1}}$$

The interval of convergence is $(-2, 2)$.

Example 4

Find a power series representation for $f(x) = \ln(5 - x)$ and its radius of convergence.

Notice that $f'(x) = \dfrac{-1}{5 - x}$. Integrating both sides of this equation gives

$$\int f'(x)\,dx = f(x) = \int \frac{-1}{5-x}\,dx.$$

We, therefore, need a power series representation for $-1/(5 - x)$ that we will integrate term-by-term to obtain the desired series.

$$\frac{-1}{5-x} = \frac{-1}{5\left(1 - \frac{x}{5}\right)}$$

Example 4 (continued)

To obtain the series for $-1/(5-x)$, replace x by $x/5$ in the series for $1/(1-x)$, and also multiply it by $-1/5$:

$$\frac{-1}{5-x} = \frac{-1}{5\left(1-\frac{x}{5}\right)} = -\frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n = \sum_{n=0}^{\infty} -\frac{x^n}{5^{n+1}}.$$

This series converges for $|x/5| < 1$, which is the same as $|x| < 5$, or $-5 < x < 5$. The radius of convergence is $R = 5$.

Integrating term-by-term:

$$\ln(5-x) = \int \frac{-1}{5-x}\,dx = \int \left(\sum_{n=0}^{\infty} -\frac{x^n}{5^{n+1}}\right) dx = \sum_{n=0}^{\infty} \int \left(-\frac{x^n}{5^{n+1}}\,dx\right)$$

Example 4 (continued)

$$\ln(5-x) = \sum_{n=0}^{\infty} \int \left(-\frac{x^n}{5^{n+1}}\, dx \right) = C + \sum_{n=0}^{\infty} -\frac{x^{n+1}}{(n+1)5^{n+1}}$$

To find C, set $x = 0$:

$$\ln(5-0) = \ln(5) = C + \sum_{n=0}^{\infty} -\frac{0^{n+1}}{(n+1)5^{n+1}} = C.$$

Example 4 (continued)

Therefore,

$$\begin{aligned} \ln(5-x) &= \ln(5) + \sum_{n=0}^{\infty} -\frac{x^{n+1}}{(n+1)5^{n+1}} \\ &= \ln(5) - \frac{x}{5} - \frac{x^2}{2\cdot 5^2} - \frac{x^3}{3\cdot 5^3} - \frac{x^4}{4\cdot 5^4} - \cdots \end{aligned}$$

The radius of convergence is the same as for the original series: $R = 5$. The interval of convergence is again $(-5, 5)$.

Example 5

Evaluate $\int \frac{t}{1-t^8}\,dt$ as a power series.

We use a familiar strategy: obtain a power series for $1/(1-t^8)$ using the power series for $1/(1-x)$, and then multiply by t:

$$\frac{t}{1-t^8} = t\left(\frac{1}{1-t^8}\right) = t\sum_{n=0}^{\infty}(t^8)^n = \sum_{n=0}^{\infty} t^{8n+1}.$$

This can be integrated term-by-term to obtain the desired result.

Example 5 (continued)

$$\begin{aligned}\int \frac{t}{1-t^8}\,dt &= \int \left(\sum_{n=0}^{\infty} t^{8n+1}\right) dt \\ &= \sum_{n=0}^{\infty} \left(\int t^{8n+1}\,dt\right) \\ &= C + \sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2}\end{aligned}$$

The geometric series for $\dfrac{1}{1-t^8}$ converges when $|t^8| < 1$, that is, when $-1 < t < 1$ (radius $R = 1$).

The series for $\displaystyle\int \frac{t}{1-t^8}\,dt$ also converges on $-1 < t < 1$.

Example 6

Find a power series representation for $f(x) = \tan^{-1}(x)$.

Note that $f'(x) = \dfrac{1}{1+x^2}$. Therefore

$$\tan^{-1}(x) = \int \frac{1}{1+x^2}\,dx.$$

We, therefore, need a power series for $1/(1+x^2)$. This can be obtained by substituting $-x^2$ for x in the power series for $1/(1-x)$:

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}.$$

Example 6 (continued)

$$\begin{aligned}
\tan^{-1}(x) &= \int \frac{1}{1+x^2}\,dx \\
&= \int \left(\sum_{n=0}^{\infty} (-1)^n x^{2n} \right) dx \\
&= \sum_{n=0}^{\infty} \int \left((-1)^n x^{2n}\,dx \right) \\
&= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \\
&= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots
\end{aligned}$$

Example 6 (continued)

To find C, set $x = 0$. This leads to $C = \tan^{-1}(0) = 0$. Therefore

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

The series for $1/(1+x^2)$ converges when $|-x^2| < 1$ which is the same as $-1 < x < 1$.

Therefore, the series for $\tan^{-1}(x)$ also converges when $-1 < x < 1$. In addition, it can be shown that it converges when $x = \pm 1$ (much harder to do).

If we substitute $x = 1$ into the series above we get:

$$\tan^{-1}(1) = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$