Question 1:

(a) [3 points] Let $f(x) = \arcsin(\sinh(x))$. Evaluate $f'(0)$.

$$f'(x) = \frac{1}{\sqrt{1-\sinh^2(x)}} \cdot \cosh(x)$$

$$f'(0) = \frac{1}{\sqrt{1-0^2}} \cdot 1 = \boxed{1}$$

(b) [4 points] Evaluate

$$\lim_{x\to 0} \frac{e^{x^2}-1}{x\sin x} \sim \frac{0}{0}$$

$$\overset{H}{=} \lim_{x\to 0} \frac{2xe^{x^2}}{\sin x + x\cos x} \sim \frac{0}{0}$$

$$\overset{H}{=} \lim_{x\to 0} \frac{2e^{x^2} + (2x)^2 e^{x^2}}{\cos x + \cos x - x\sin x}$$

$$= \frac{2}{1+1}$$

$$= \boxed{1}$$

(c) [5 points] Evaluate

$$L = \lim_{x\to 0} (\cos x)^{\frac{1}{x^2}} \sim 1^{\infty}$$

$$= \lim_{x\to 0} e^{\frac{1}{x^2}\ln(\cos x)}.$$

$$\lim_{x\to 0} \frac{\ln(\cos x)}{x^2} \sim \frac{0}{0}$$

$$\boxed{\therefore L = e^{-\frac{1}{2}}}$$

$$\overset{H}{=} \lim_{x\to 0} \frac{\frac{-\sin x}{\cos x}}{2x} \sim \frac{0}{0}$$

$$\overset{H}{=} \lim_{x\to 0} \frac{\frac{-\cos^2 x - \sin^2 x}{\cos^2 x}}{2} = \frac{-1}{2}$$

Question 2:

(a) [**4 points**] A particle is moving with acceleration $a(t) = t + \sin t$ m/s^2. If velocity at time $t = 0$ is $v(0) = 2$ m/s and initial position is $s(0) = 0$ m, determine $s(t)$, the position of the particle at time $t =$ seconds.

$$v(t) = \frac{t^2}{2} - \cos t + C_1$$

$$v(0) = 2 \Rightarrow \frac{0^2}{2} - \cancel{\cos(0)} + C_1 = 2, \quad \therefore C_1 = 3$$

$$\therefore v(t) = \frac{t^2}{2} - \cos(t) + 3$$

$$\therefore s(t) = \frac{t^3}{6} - \sin(t) + 3t + C_2$$

$$s(0) = 0 \Rightarrow \cancel{\frac{0^3}{6}} - \cancel{\sin(0)} + \cancel{3(0)} + C_2 = 0, \quad \therefore C_2$$

$$\therefore \boxed{s(t) = \frac{t^3}{6} - \sin(t) + 3t}$$

(b) [**3 points**] The average value of $f(x) = x^3$ over the interval $[0, a]$ is 16. Determine the value of a.

$$\frac{1}{a-0} \int_0^a x^3 \, dx = 16.$$

$$\frac{1}{a} \left[\frac{x^4}{4}\right]_0^a = 16$$

$$\frac{1}{\cancel{a}} \cdot \frac{a^{\cancel{4}\,3}}{4} = 16$$

$$a^3 = 64$$

$$\boxed{a = 4}$$

(c) [**3 points**] Use the Fundamental Theorem of Calculus to determine $f(x)$ if

$$\int_0^x f(t)\, dt = xe^{x^2} + 1 .$$

$$\frac{d}{dx}\left[\int_0^x f(t)\, dt\right] = \frac{d}{dx}\left[xe^{x^2} + 1\right]$$

$$f(x) = 1 \cdot e^{x^2} + x \cdot e^{x^2} \cdot (2x)$$

$$\boxed{f(x) = e^{x^2} + 2x^2 e^{x^2}}$$

Question 3:

(a) [**3 points**] Evaluate:

$$\int \frac{2x^3 + 5\sqrt{x} - 3}{x}\, dx$$

$$= 2\int x^2\, dx + 5\int x^{-1/2}\, dx - 3\int \frac{1}{x}\, dx$$

$$= \frac{2x^3}{3} + \frac{5x^{1/2}}{1/2} - 3\ln|x| + C$$

$$= \boxed{\frac{2}{3}x^3 + 10x^{1/2} - 3\ln|x| + C}$$

(b) [**3 points**] Evaluate:

$$I = 2\int \frac{\sin(\sqrt{x}+1)}{2\sqrt{x}}\, dx$$

Let $u = \sqrt{x}+1$, $du = \frac{1}{2\sqrt{x}}\, dx$

$$\therefore\ I = 2\int \sin(u)\, du$$

$$= -2\cos(u) + C$$

$$= \boxed{-2\cos(\sqrt{x}+1) + C}$$

(c) [**4 points**] The population is growing at a rate of $P'(t) = 200e^{t/5}$ individuals per year, where year $t = 0$ corresponds to the present. If the current population size is $P(0) = 1000$, determine the population size at the end of five years.

$$P(5) - P(0) = \int_0^5 P'(t)\, dt$$

$$\therefore\ P(5) = 1000 + \int_0^5 200e^{\frac{t}{5}}\, dt$$

$$= 1000 + \frac{200}{(1/5)}\left[e^{\frac{t}{5}}\right]_0^5$$

$$= 1000 + 1000\left(e^1 - e^0\right)$$

$$= 1000\,(1 + e - 1) = \boxed{1000e \text{ individuals.}}$$

Question 4:

(a) [6 points] Evaluate:

$$I = \int (x^2+1)e^{-x}\,dx$$

$u = x^2+1$; $dv = e^{-x}dx$

$du = 2x\,dx$; $v = -e^{-x}$

$$\therefore I = -(x^2+1)e^{-x} + \int \underbrace{2x\, e^{-x}\, dx}$$

$u = 2x$; $dv = e^{-x}dx$

$du = 2$; $v = -e^{-x}$

$$\therefore I = -(x^2+1)e^{-x} - 2xe^{-x} + \int 2e^{-x}\,dx$$

$$= \boxed{-(x^2+1)e^{-x} - 2xe^{-x} - 2e^{-x} + C}$$

(b) [6 points] Evaluate:

$$I = \int \sec^6 x \tan^2 x\,dx$$

$$= \int \sec^4 x \tan^2 x \sec^2 x\,dx$$

$$= \int (1+\tan^2 x)^2 \tan^2 x \sec^2 x\,dx$$

$u = \tan x$, $du = \sec^2 x\,dx$

$$\therefore I = \int (1+u^2)^2 u^2\,du$$

$$= \int u^2 + 2u^4 + u^6\,du$$

$$= \frac{u^3}{3} + \frac{2u^5}{5} + \frac{u^7}{7} + C$$

$$= \boxed{\frac{\tan^3 x}{3} + \frac{2\tan^5 x}{5} + \frac{\tan^7 x}{7} + C}$$

Question 5 [8 points]: Evaluate:

$$I = \int x^3\sqrt{4-x^2}\,dx$$

$x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$

$$\therefore\ I = \int 8\sin^3\theta\,\sqrt{4-4\sin^2\theta}\;2\cos\theta\,d\theta$$

$$= 32\int \sin^3\theta\,\cos^2\theta\,d\theta$$

$$= 32\int (1-\cos^2\theta)\cos^2\theta\,\sin\theta\,d\theta$$

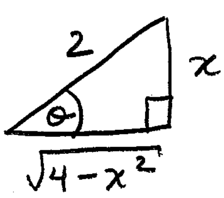
$u = \cos\theta$, $du = -\sin\theta\,d\theta$

$$\therefore\ I = -32\int (1-u^2)u^2\,du$$

$$= -32\int u^2 - u^4\,du$$

$$= -32\left[\frac{u^3}{3} - \frac{u^5}{5}\right] + C$$

$$= -32\left[\frac{\cos^3\theta}{3} - \frac{\cos^5\theta}{5}\right] + C$$



$$\therefore\ I = -32\left[\frac{1}{3}\left(\frac{\sqrt{4-x^2}}{2}\right)^3 - \frac{1}{5}\left(\frac{\sqrt{4-x^2}}{2}\right)^5\right] + C$$

$$= \boxed{-\frac{4}{3}(4-x^2)^{3/2} + \frac{1}{5}(4-x^2)^{5/2} + C}$$

Question 6 [8 points]: Evaluate:

$$I = \int_0^1 \frac{4x^2+x+3}{(x+1)(x^2+1)}\,dx$$

$$\frac{4x^2+x+3}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$= \frac{Ax^2+A + Bx^2+Cx+Bx+C}{(x+1)(x^2+1)}$$

$$= \frac{(A+B)x^2 + (C+B)x + A+C}{(x+1)(x^2+1)}$$

$\therefore\ A+B = 4 \Rightarrow B = 4-A$

$C+B = 1 \Rightarrow C+4-A = 1 \Rightarrow C = A-3$

$A+C = 3 \Rightarrow A+(A-3) = 3 \Rightarrow 2A = 6 \Rightarrow A = 3$

$\therefore\ C = 3-3 =$

$\therefore\ B = 4-3 =$

$$\therefore\ I = \int_0^1 \frac{3}{x+1} + \frac{x}{x^2+1}\,dx$$

$$= 3\int_0^1 \frac{1}{x+1}\,dx + \frac{1}{2}\int_0^1 \frac{2x}{x^2+1}\,dx$$

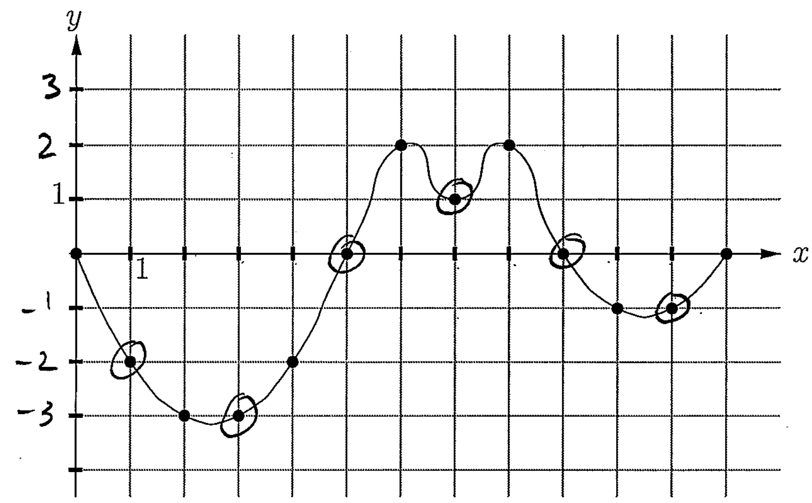
$$= 3\Big[\ln|x+1|\Big]_0^1 + \frac{1}{2}\Big[\ln|x^2+1|\Big]_0^1$$

$$= 3\left(\ln(2) - \cancelto{0}{\ln(1)}\right) + \frac{1}{2}\left(\ln(2) - \cancelto{0}{\ln(1)}\right)$$

$$= \boxed{\frac{7}{2}\ln 2}$$

Question 7:

(a) [4 points] Use the graph of $y = f(x)$ below to approximate $\int_0^{12} f(x)\,dx$ using M_6, the midpoint rule on six subintervals.



$$\Delta x = \frac{12}{6} = 2$$

$$\therefore \int_0^{12} f(x)\,dx \approx M_6 = 2\left[(-2)+(-3)+0+1+0+(-1)\right]$$

$$= \boxed{-10}$$

(b) [4 points] If Simpson's Rule is used to approximate $\int_0^{180} \left(\frac{x^4+x}{24}\right) dx$, how many subintervals are required to guarantee an error of at most 1? Recall, the error in using Simpson's rule to approximate $\int_a^b f(x)\,dx$ is at most $\frac{K(b-a)^5}{180n^4}$, where $|f^{(4)}(x)| \le K$ on $[a, b]$.

$$f(x) = \frac{x^4+x}{24}$$

$$f'(x) = \frac{4x^3}{24}$$

$$f''(x) = \frac{12x^2}{24}$$

$$f'''(x) = \frac{24x}{24}$$

$$f^{(iv)}(x) = 1$$

$$\therefore K = 1$$

$$\therefore \frac{K(b-a)^5}{180n^4} \le 1$$

$$\frac{1\cdot(180-0)^5}{180n^4} \le 1$$

$$180^4 \le n^4$$

$$180 \le n$$

$$\therefore \boxed{n \ge 180}$$

Question 8:

(a) [**3 points**] Use the comparison theorem to determine if the integral converges or diverges (do not attempt to evaluate the integral):

$$\int_1^\infty \frac{e^{-x}}{x^4+1}\,dx$$

$$0 \le \frac{e^{-x}}{x^4+1} \le \frac{1}{x^4} \quad \text{on } [1,\infty).$$

Since $\int_1^\infty \frac{1}{x^4}\,dx$ converges, so

must $\int_1^\infty \frac{e^{-x}}{x^4+1}\,dx$.

(b) [**5 points**] Evaluate the improper integral:

$$\int_0^1 \frac{e^x}{\sqrt{e^x-1}}\,dx$$

For $I = \int \frac{e^x}{\sqrt{e^x-1}}\,dx$, let $u = e^x - 1$, $du = e^x\,dx$

$\therefore I = \int u^{-\frac{1}{2}}\,du = 2u^{\frac{1}{2}} + C = 2\sqrt{e^x-1} + C$

$$\therefore \int_0^1 \frac{e^x}{\sqrt{e^x-1}}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{e^x}{\sqrt{e^x-1}}\,dx$$

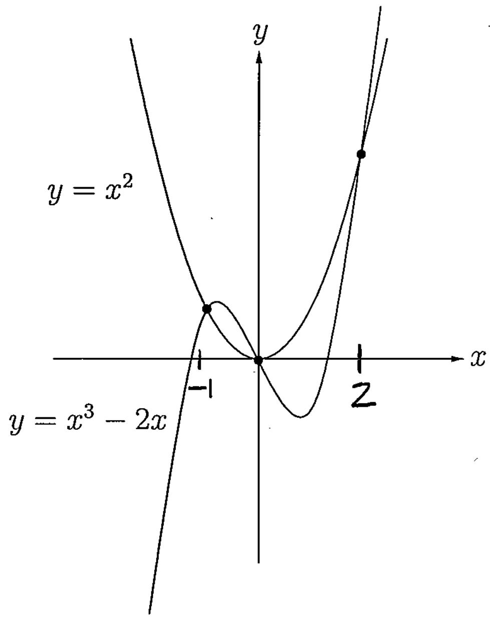
$$= \lim_{t\to 0^+} \left[2\sqrt{e^x-1}\right]_t^1$$

$$= \lim_{t\to 0^+} \left(2\sqrt{e-1} - 2\sqrt{e^t-1}\right)$$

$$= \boxed{2\sqrt{e-1}}$$

Question 9:

(a) [5 points] Determine the area of the region bounded between the curves $y = x^2$ and $y = x^3 - 2x$



$$x^2 = x^3 - 2x$$
$$x^3 - x^2 - 2x = 0$$
$$x(x^2 - x - 2) = 0$$
$$x(x+1)(x-2) = 0$$
$$\therefore x = 0,\ x = -1,\ x = 2$$

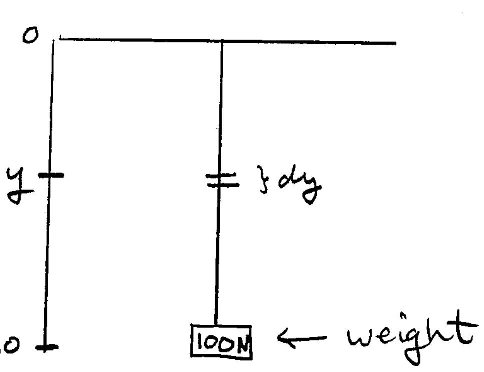
$$\therefore A = \int_{-1}^{0} x^3 - 2x - x^2\,dx + \int_0^2 x^2 - x^3 + 2x\,dx$$

$$= \left[\frac{x^4}{4} - x^2 - \frac{x^3}{3}\right]_{-1}^{0} + \left[\frac{x^3}{3} - \frac{x^4}{4} + x^2\right]_0^2$$

$$= \left(\frac{0^4}{4} - 0^2 - \frac{0^3}{3}\right) - \left(\frac{1}{4} - 1 + \frac{1}{3}\right) + \left(\frac{8}{3} - \cancel{\frac{16}{4}} + \cancel{4}\right) - \left(\frac{0^3}{3} - \frac{0^4}{4}\right)$$

$$= \frac{5}{12} + \frac{8}{3} = \frac{5 + 32}{12} = \boxed{\frac{37}{12}}$$

(b) [5 points] A 10 m chain with a 100 N weight at the end hangs over the side of a bridge. The weight is initially 20 m above the ground, and one metre of chain weighs 10 N. A person pulls the chain and weight up onto the bridge deck. How much work is done?



$$W_{weight} = F \cdot d = (100\text{ N})(10\text{ m})$$
$$= 1000\text{ J}.$$

$$W_{chain} = \int_0^{10} (y\text{ m})\left(10\ \frac{\text{N}}{\cancel{\text{m}}}\right)(dy\ \cancel{\text{m}})$$

$$= 10 \int_0^{10} y\,dy$$

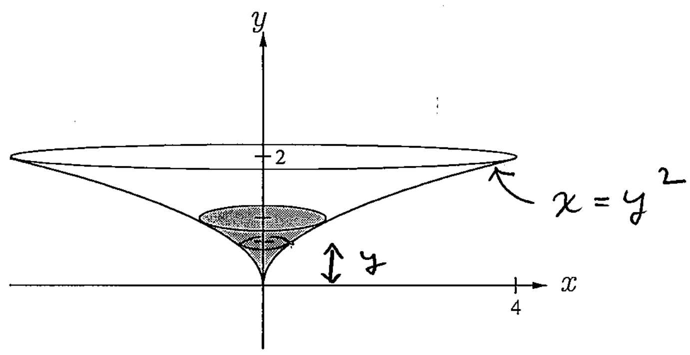
$$= \frac{10}{2}\left[y^2\right]_0^{10}$$

$$= 500\text{ J}$$

$$\therefore \text{Total work} = 1000 + 500 = \boxed{1500\text{ J}}$$

Question 10

(a) [**7 points**] A vessel is formed by rotating the curve $y = \sqrt{x}$ about the y-axis as shown below. The vessel has a top radius of 4 m and a depth of 2 m. If the vessel is initially filled with water to a depth of 1 m, how much work is required empty the vessel by pumping the water up and over the top rim? Recall that the density of water is $\rho = 1000$ kg/m^3 and acceleration due to gravity is $g = 9.8$ m/s^2. (You may leave your final answer as a product of fractions.)



$$W = \int_0^1 9.8(1000)\,\pi (y^2)^2 (2-y)\,dy$$

$$= 9800\pi \int_0^1 2y^4 - y^5\,dy$$

$$= 9800\pi \left(\frac{2}{5}\left[y^5\right]_0^1 - \frac{1}{6}\left[y^6\right]_0^1\right)$$

$$= 9800\pi \left(\frac{2}{5} - \frac{1}{6}\right)$$

$$= 9800\pi \left(\frac{12-5}{30}\right)$$

$$= 9800\pi \left(\frac{7}{30}\right)$$

$$= \boxed{980\pi \left(\frac{7}{3}\right) \text{ J}}$$

(b) [**3 points**] Determine the total volume of the vessel described in part (a). (Disks would be best here.)

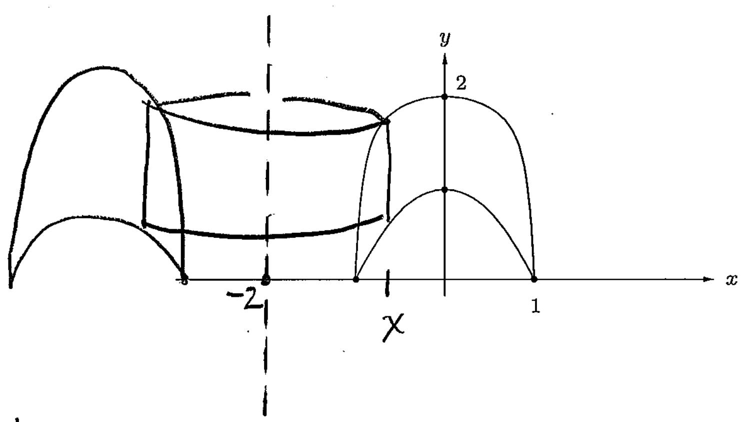
$$V = \int_0^2 \pi (y^2)^2\,dy$$

$$= \frac{\pi}{5}\left[y^5\right]_0^2$$

$$= \boxed{\frac{32\pi}{5}}$$

Question 11:

(a) [**3 points**] The region bounded between the curves $y = 1 - x^2$ and $y = 2(1 - x^4)$ is rotated about the line $x = -2$. Set up the integral representing the volume of the resulting solid. DO NOT EVALUATE THE INTEGRAL. (Cylindrical shells would be best here.)

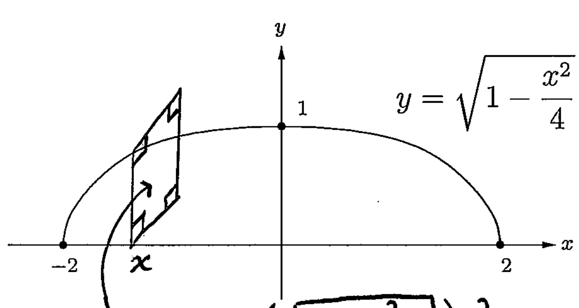


$$V = \int_{-1}^{1} 2\pi (x+2)\left(2(1-x^4) - (1-x^2)\right) dx$$

(b) [**3 points**] Referring to the diagram for part (a), the same bounded region is rotated about the horizontal line $y = 2$. Set up the integral representing the volume of the resulting solid. DO NOT EVALUATE THE INTEGRAL. (Disks would be best here.)

$$V = \int_{-1}^{1} \pi \left[(2-(1-x^2))^2 - (2 - 2(1-x^4))^2\right] dx$$

(c) [**4 points**] The base (flat bottom surface) of the solid S is the region between the curve $y = \sqrt{1 - \frac{x^2}{4}}$ and the x-axis. Cross-sections perpendicular to the x-axis are squares. Determine the volume of S.



$$A(x) = \left(\sqrt{1 - \frac{x^2}{4}}\right)^2 = 1 - \frac{x^2}{4}$$

$$\therefore V = \int_{-2}^{2} 1 - \frac{x^2}{4}\, dx = 2\int_0^2 1 - \frac{1}{4}x^2\, dx$$

$$= 2\left[x - \frac{1}{12}x^3\right]_0^2$$

$$= 2\left[2 - \frac{8}{12}\right] = \boxed{\frac{8}{3}}$$

Question 12:

(a) [**4 points**] Solve the differential equation:

$$\frac{dy}{dx} = \frac{y\cos x}{1+y^2} \qquad y(0) = 1$$

$$\int \frac{1+y^2}{y}\,dy = \int \cos x\,dx$$

$$\int \frac{1}{y} + y\,dy = \int \cos x\,dx$$

$$\ln|y| + \frac{y^2}{2} = \sin x + C$$

$$y(0)=1 \Rightarrow \cancel{\ln|1|}^{0} + \frac{1^2}{2} = \cancel{\sin(0)}^{0} + C$$

$$\Rightarrow C = \frac{1}{2}$$

$$\therefore \boxed{\ln|y| + \frac{y^2}{2} = \sin x + \frac{1}{2}}$$

(b) [**6 points**] A tank contains 1000 L of pure water. Salt water with a concentration of 0.1 kg/L of salt enters the tank at a rate of 10 L per minute, while water is pumped out of the bottom of the tank at the same rate. The water in the tank is kept thoroughly mixed. The mass $A(t)$ of dissolved salt in the tank at time t is modeled by the differential equation

$$\frac{dA}{dt} = 1 - \frac{1}{100}A \qquad A(0) = 0$$

Solve this differential equation for $A(t)$.

$$\int \frac{1}{1-\frac{1}{100}A}\,dA = \int dt$$

$$-100\ln\left|1-\frac{1}{100}A\right| = t + C_1$$

$$\ln\left|1-\frac{1}{100}A\right| = -\frac{1}{100}t + C_2$$

$$\left|1-\frac{1}{100}A\right| = C_3 e^{-\frac{1}{100}t}$$

$$1-\frac{1}{100}A = C_4 e^{-\frac{1}{100}t}$$

$$A = 100 - C_5 e^{-\frac{1}{100}t}$$

$A(0) = 0$, so

$$0 = 100 - C_5 e^0$$

$$\therefore C_5 = 100.$$

$$\therefore \boxed{A(t) = 100 - 100e^{-\frac{1}{100}t}}$$

Question 13:

(a) **[4 points]** Use the definition to construct $T_2(x)$, the Maclaurin series of degree 2, for the function $f(x) = \sin(x^2 + x)$

$f(x) = \sin(x^2+x)$; $f(0) = \sin(0^2+0) = 0$

$f'(x) = \cos(x^2+x)(2x+1)$; $f'(0) = \cos(0^2+0)(2\cdot 0+1) =$

$f''(x) = -\sin(x^2+x)(2x+1)^2 + \cos(x^2+x)\cdot 2$; $f''(0) = 0 + 2 =$

$$\therefore T_2(x) = f(0) + f'(0)\,x + f''(0)\,\frac{x^2}{2}$$

$$= 0 + 1\cdot x + 2\cdot\frac{x^2}{2}$$

$$= \boxed{x + x^2}$$

(b) **[4 points]** Use Maclaurin series (not L'Hospital's Rule) to evaluate the limit

$$\lim_{x\to 0}\frac{\cos x - 1}{x\sin x}$$

$$= \lim_{x\to 0}\frac{\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots\right) - 1}{x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}$$

$$= \lim_{x\to 0}\frac{-\frac{x^2}{2} + \frac{x^4}{4!} - \cdots}{x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \cdots}$$

$$= \lim_{x\to 0}\frac{-\frac{1}{2} + \frac{x^2}{4!} - \cdots}{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots}$$

$$= \boxed{-\frac{1}{2}}$$