Question 1: (Integration by Parts) Determine $\int e^{2x}\sin(x)\,dx$ $= I$

let $u = e^{2x}$ $dv = \sin(x)\,dx$

$du = 2e^{2x}dx$ $v = -\cos(x)$

$$I = \int u\,dv$$
$$= uv - \int v\,du$$
$$= -e^{2x}\cos(x) + 2\underbrace{\int \cos(x)e^{2x}dx}$$

$u = e^{2x}$ $dv = \cos(x)$

$du = 2e^{2x}dx$ $v = \sin(x)$

$$= -e^{2x}\cos(x) + 2\left[e^{2x}\sin(x) - 2\int \sin(x)e^{2x}dx\right]$$
$$\therefore I = -e^{2x}\cos(x) + 2e^{2x}\sin(x) - 4I$$
$$\therefore I = \frac{1}{5}e^{2x}\left[2\sin(x) - \cos(x)\right] + C$$

[5]

Question 2: (Trigonometric Integral) Determine $\int t\sin^2(t)\,dt$ $= I$

$$I = \int t\left(\frac{1-\cos(2t)}{2}\right)dt$$
$$= \frac{1}{2}\int t\,dt - \frac{1}{2}\underbrace{\int t\cos(2t)\,dt}$$

$u = t$ $dv = \cos(2t)\,dt$

$du = dt$ $v = \frac{1}{2}\sin(2t)$

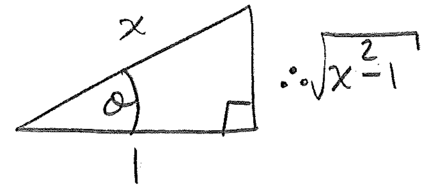
$$= \frac{1}{2}\left(\frac{t^2}{2}\right) - \frac{1}{2}\left[t\cdot\frac{1}{2}\sin(2t) - \int \frac{1}{2}\sin(2t)\,dt\right]$$
$$= \boxed{\frac{t^2}{4} - \frac{t\sin(2t)}{4} - \frac{\cos(2t)}{8} + C}$$

[5]

Question 3: (Trigonometric Substitution) Determine $\int \frac{1}{x^3\sqrt{x^2-1}}\,dx = I$

let $x = \sec\theta$

$dx = \sec\theta\tan\theta\,d\theta$



$$I = \int \frac{1\cdot\sec\theta\tan\theta}{\sec^3\theta\sqrt{\sec^2\theta-1}}\,d\theta$$

$$= \int \frac{\cancel{\sec\theta}\ \cancel{\tan\theta}}{\sec^{\cancel{3}\,2}\theta\ \cancel{\tan\theta}}\,d\theta$$

$$= \int \cos^2\theta\,d\theta$$

$$= \int \frac{1+\cos(2\theta)}{2}\,d\theta$$

$$= \frac{\theta}{2} + \frac{\sin(2\theta)}{4} + C$$

$$= \frac{\theta}{2} + \frac{\cancel{2}\sin\theta\cos\theta}{2\ \cancel{4}} + C$$

$$= \frac{\operatorname{arcsec}(x)}{2} + \frac{1}{2}\frac{\sqrt{x^2-1}}{x}\cdot\frac{1}{x} + C$$

$$= \boxed{\frac{1}{2}\left(\operatorname{arcsec}(x) + \frac{\sqrt{x^2-1}}{x^2}\right) + C}$$

[10]

Question 4: (Partial Fractions) Determine $\int \frac{25x}{(x+2)^2(x-3)}\,dx = I$

$$\frac{25x}{(x+2)^2(x-3)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-3}$$

$$= \frac{A(x+2)(x-3) + B(x-3) + C(x+2)^2}{(x+2)^2(x-3)}$$

$$= \frac{(A+C)x^2 + (-A+B+4C)x + (-6A-3B+4C)}{(x+2)^2(x-3)}$$

$A + C = 0$ ①
$-A + B + 4C = 25$ ②
$-6A - 3B + 4C = 0$ ③

① $\Rightarrow$ $C = -A$
② $\Rightarrow$ $B = 25 + A - 4C$
$= 25 + A - 4(-A)$
$= 25 + 5A$

③ $\Rightarrow$ $-6A - 3(25 + 5A) + 4(-A) = 0$
$-25A = 75$
$\therefore\ A = -3$
$B = 25 + 5(-3) = 10$
$C = -(-3) = 3.$

$$\therefore I = \int \frac{-3}{x+2} + \frac{10}{(x+2)^2} + \frac{3}{x-3}\,dx$$

$$= \boxed{-3\ln|x+2| - \frac{10}{x+2} + 3\ln|x-3| + C}$$

[10]

Question 5: Determine $\displaystyle\int \frac{x^3+4}{x^2+4}\,dx$ (Hint: long division)

$$\begin{array}{r} x \\ x^2+0x+4 \,\overline{)\, x^3+0x^2+0x+4} \\ -(x^3+0x^2+4x) \downarrow \\ \hline -4x+4 \end{array}$$

$$\therefore I = \int x + \frac{-4x+4}{x^2+4}\,dx$$

$$= \int x\,dx - 2\underbrace{\int \frac{2x}{x^2+4}\,dx}_{\substack{u=x^2+4 \\ du=2x\,dx}} + \underbrace{4\int \frac{1}{x^2+2^2}\,dx}_{\text{Formula \#18}}$$

$$= \boxed{\frac{x^2}{2} - 2\ln|x^2+4| + 2\arctan\left(\frac{x}{2}\right) + C}$$

[5]

Question 6: If the Midpoint Rule is used to approximate $\displaystyle\int_1^4 \frac{1}{\sqrt{x}}\,dx$, how many subintervals n are required to guarantee that the error in the approximation is at most $1/96$? State your answer as a natural number, and make correct use of inequality notation.

$f(x) = x^{-1/2}$, $[a,b] = [1,4]$

$f'(x) = -\frac{1}{2}x^{-3/2}$

$f''(x) = \frac{3}{4}x^{-5/2}$.

On $[1,4]$,

$|f''(x)| = \left|\frac{3}{4x^{5/2}}\right| \leqslant \frac{3}{4}$,

So $K = \frac{3}{4}$

We want $\dfrac{K(b-a)^3}{24n^2} \leqslant \dfrac{1}{96}$

$$\frac{(3/4)(4-1)^3}{24n^2} \leqslant \frac{1}{96}$$

$$\frac{3^4}{\cancel{96}n^2} \leqslant \frac{1}{\cancel{96}}$$

$$3^4 \leqslant n^2$$

$$3^2 \leqslant n$$

$$\therefore \boxed{n \geqslant 9}$$

[5]

Question 7: Determine if the improper integral $\int_{-\infty}^{\infty} \frac{x^2}{9+x^3}\,dx$ converges or diverges. If it converges give the value, if it diverges then say so. Make proper use of any required limits and use proper notation.

$$I = \int_{-\infty}^{\infty} \frac{x^2}{9+x^3}\,dx = \underbrace{\int_{-\infty}^{0} \frac{x^2}{9+x^3}\,dx}_{} + \underbrace{\int_{0}^{\infty} \frac{x^2}{9+x^3}\,dx}_{}$$

$$= I_1 + I_2$$

For I_2:

$$I_2 = \lim_{b\to\infty} \int_0^b \frac{x^2}{9+x^3}\,dx \quad \begin{cases} u = 9+x^3 \\ du = 3x^2\,dx \end{cases}$$

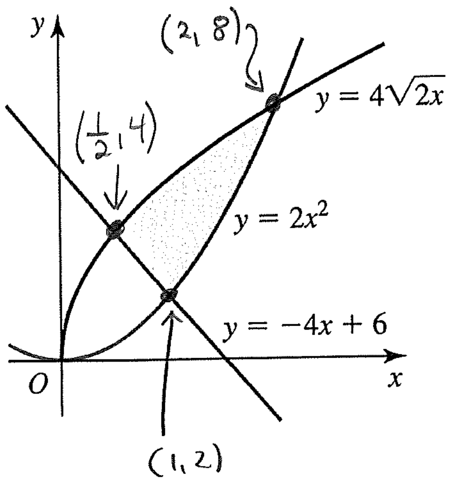
$$= \lim_{b\to\infty} \frac{1}{3}\Big[\ln|9+x^3|\Big]_0^b$$

$$= \lim_{b\to\infty} \frac{1}{3}\left[\ln|9+b^3| - \ln|9+0^3|\right]$$

$$= \infty.$$

Since I_2 diverges, I diverges also. **[5]**

Question 8: Express the shaded area as an integral, or the sum of several integrals. It is not necessary to evaluate the integrals.



$$\boxed{A = \int_{\frac{1}{2}}^{1} \left[4\sqrt{2x} - (-4x+6)\right]dx + \int_1^2 \left[4\sqrt{2x} - 2x^2\right]dx.}$$

[5]