Question 1:

(a) Find the most general antiderivative:

(i) $f(x) = \sqrt{x}(x-1) - \frac{\csc^2(x)}{\pi}$ $= x^{3/2} - x^{1/2} - \frac{1}{\pi}\csc^2(x)$

$$\boxed{F(x) = \frac{2}{5}x^{5/2} - \frac{2}{3}x^{3/2} + \frac{1}{\pi}\cot(x) + C}$$

[2]

(ii) $f(x) = \frac{4}{x} - \frac{3}{x^4} - e^2$ $= \frac{4}{x} - 3x^{-4} - e^2$

$$F(x) = 4\ln|x| - 3\frac{x^{-3}}{-3} - e^2 x + C$$

$$= \boxed{4\ln|x| + \frac{1}{x^3} - e^2 x + C}$$

[2]

(b) A particle starts at $(0,0)$ at time $t = 0$ and heads out along the x-axis with initial velocity of $s'(0) = 10$ m/s. If the particle has acceleration at time t given by $s''(t) = -t$ m/s^2, what is the particle's displacement when the velocity is 2 m/s?

$$s''(t) = -t \Rightarrow s'(t) = -\frac{t^2}{2} + C_1$$

$$s'(0) = 10 \Rightarrow -\frac{0^2}{2} + C_1 = 10 \Rightarrow C_1 = 10$$

$$\therefore s'(t) = -\frac{t^2}{2} + 10.$$

$$s'(t) = 2 \Rightarrow -\frac{t^2}{2} + 10 = 2 \Rightarrow t = 4, \cancel{-4}$$

$$s(t) = -\frac{t^3}{6} + 10t + C_2$$

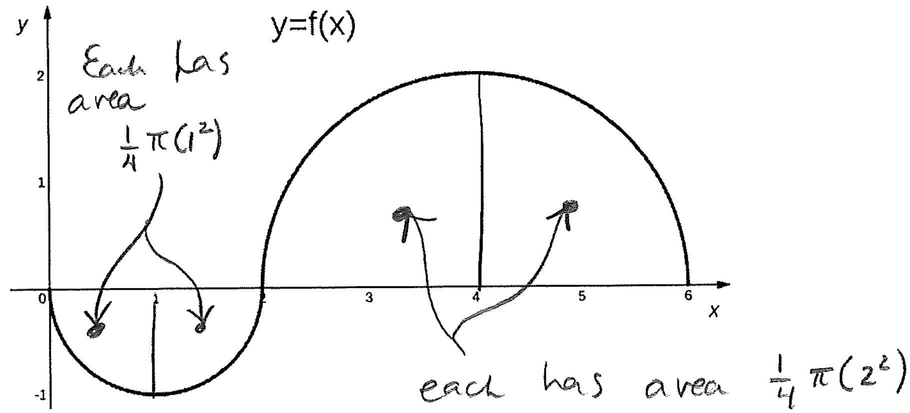
$$s(0) = 0 \Rightarrow C_2 = 0.$$

$\therefore$ when velocity is $2\frac{m}{s}$, $t = 4$, and displacement is $s(4) = -\frac{4^3}{6} + 10(4)$

$$= \boxed{\frac{88}{3} \text{ m}}$$

[6]

Question 2: For this question use the following graph of $y = f(x)$. Each of the two parts of the graph are semicircles:



(a) Compute $\int_1^4 f(x)\,dx$ $= -\frac{1}{4}\pi(1^2) + \frac{1}{4}\pi(2^2)$

$= \boxed{\frac{3}{4}\pi}$

[2]

(b) Compute $\int_0^6 2f(x)\,dx$ $= 2\int_0^6 f(x)\,dx$

$= 2\left[-(2)\left(\frac{1}{4}\pi\cdot 1^2\right) + (2)\left(\frac{1}{4}\pi\cdot 2^2\right)\right]$

$= \boxed{3\pi}$

[2]

(c) Compute $\int_1^6 (2 + |f(x)|)\,dx$

$= \int_1^6 2\,dx + \int_1^6 |f(x)|\,dx$

$= [2x]_1^6 + \left(\frac{1}{4}\pi\cdot 1^2 + 2\cdot\frac{1}{4}\pi\cdot 2^2\right)$

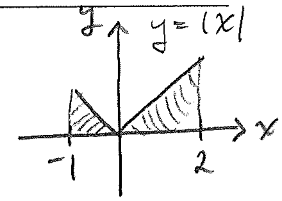
$= (12-2) + \frac{9\pi}{4}$

$= 10 + \frac{9\pi}{4}$

$= \boxed{\frac{40+9\pi}{4}}$

[2]

Question 3: Evaluate $\int_{-1}^2 (x - 2|x|)\,dx$ $= \int_{-1}^2 x\,dx - 2\int_{-1}^2 |x|\,dx$



$= \left[\frac{x^2}{2}\right]_{-1}^2 - 2\left(\frac{1}{2} + 2\right)$

$= \frac{2^2}{2} - \frac{(-1)^2}{2} - 2\left(\frac{5}{2}\right)$

$= \boxed{-\frac{7}{2}}$

[4]

Question 4: For this question let $f(x) = \int_0^{\sin(x)} \sqrt{1+t^2}\, dt$ and $g(u) = \int_1^u f(x)\, dx$.

(a) Compute $f'(0)$

$f'(x) = \sqrt{1+\sin^2(x)} \cdot \cos(x)$ by FTC I

$\therefore f'(0) = \sqrt{1+\sin^2(0)} \cdot \cos(0)$

$0 \quad 1$

$= \boxed{1}$

[2]

(b) Compute $g''(\pi/6)$

$g'(u) = f(u)$

$g''(u) = f'(u) = \sqrt{1+\sin^2(u)}\ \cos(u)$

$\therefore g''(\frac{\pi}{6}) = \sqrt{1+\sin^2(\frac{\pi}{6})} \cdot \cos(\frac{\pi}{6})$

$= \sqrt{1+(\frac{1}{2})^2} \cdot \frac{\sqrt{3}}{2} = (\frac{\sqrt{5}}{2})(\frac{\sqrt{3}}{2}) = \boxed{\frac{\sqrt{15}}{4}}$

[3]

Question 5: Determine the value of the positive real number k if the average value of $f(x) = x^2 - x$ over the interval $[0, k]$ is k.

$\frac{1}{k-0} \int_0^k x^2 - x\, dx = k$

$\Rightarrow \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^k = k^2$

$\Rightarrow \frac{k^3}{3} - \frac{k^2}{2} = k^2$

$\Rightarrow k^2\left(\frac{k}{3} - \frac{3}{2}\right) = 0$

$\Rightarrow \frac{k}{3} - \frac{3}{2} = 0$ since $k \neq 0$

$\Rightarrow \boxed{k = \frac{9}{2}}$

[5]

Question 6: Evaluate the following definite integrals:

(a) $\int_1^4 \left(\frac{x}{2}+\frac{2}{x}\right)\,dx$ $= \left[\frac{1}{2}\frac{x^2}{2}+2\ln|x|\right]_1^4$

$= \left(\frac{4^2}{4}+2\ln|4|\right)-\left(\frac{1^2}{4}+\cancel{2\ln|1|}^{\,0}\right)$

$= \boxed{\frac{15}{4}+2\ln|4|}$

[2]

(b) $\int_0^{\pi/4}\left(\sqrt{2}\cos(x)-\sec^2(x)\right)\,dx$

$= \left[\sqrt{2}\sin(x)-\tan(x)\right]_0^{\frac{\pi}{4}}$

$= \left(\sqrt{2}\sin(\tfrac{\pi}{4})-\tan(\tfrac{\pi}{4})\right)-\left(\cancel{\sqrt{2}\sin(0)-\tan(0)}\right) \to 0$

$= \sqrt{2}\cdot\frac{1}{\sqrt{2}}-1 = \boxed{0}$

[3]

(c) $\int_{-1}^2 t(t-3)(t+2)\,dt$ $= \int_{-1}^2 t^3-t^2-6t\,dt$

$= \left[\frac{t^4}{4}-\frac{t^3}{3}-\frac{\cancel{6}^{3}t^2}{\cancel{2}}\right]_{-1}^2$

$= \left(\frac{2^4}{4}-\frac{2^3}{3}-3(2)^2\right)-\left(\frac{(-1)^4}{4}-\frac{(-1)^3}{3}-3(-1)^2\right)$

$= -5-3-\frac{1}{4} = \boxed{-\frac{33}{4}}$

[3]

(d) $\int_{-1000}^{1000}\frac{x^5}{1+e^{(x^2)}}\,dx$

Here $f(x)=\frac{x^5}{1+e^{x^2}}$ is <u>odd</u> since $f(-x)=\frac{(-x)^5}{1+e^{(-x)^2}}=-\frac{x^5}{1+e^{x^2}}=-f(x)$,

so $\int_{-1000}^{1000}\frac{x^5}{1+e^{x^2}}\,dx = \boxed{0}$.

[2]

Question 7: (Substitution Method) Determine the following:

(a) $\int \frac{\cos(\ln(x))}{x}\,dx = I$

Let $u = \ln(x)$, $du = \frac{1}{x}dx$

$$I = \int \cos(u)\,du = \sin(u) + C$$
$$= \boxed{\sin(\ln(x)) + C}$$

[3]

(b) $\frac{1}{5}\int \frac{5\cos(5x)}{(1+\sin(5x))^2}\,dx = I$

let $u = 1+\sin(5x)$, $du = 5\cos(5x)\,dx$

$$I = \frac{1}{5}\int u^{-2}\,du = \frac{1}{5}\frac{u^{-1}}{-1} + C$$
$$= \boxed{\frac{-1}{5(1+\sin(5x))} + C}$$

[3]

(c) $\int t(t+1)^{1/4}\,dt = I$

let $u = t+1 \Rightarrow t = u-1$

$du = dt$

$$I = \int (u-1)u^{1/4}\,du$$
$$= \int u^{\frac{5}{4}} - u^{1/4}\,du$$
$$= \frac{4}{9}u^{\frac{9}{4}} - \frac{4}{5}u^{5/4} + C$$
$$= \boxed{\frac{4}{9}(t+1)^{\frac{9}{4}} - \frac{4}{5}(t+1)^{\frac{5}{4}} + C}$$

[4]