Question 1:

(a) Use a linear approximation $T_1(x)$ for $f(x) = \dfrac{1}{\sqrt{1+x}}$ to approximate $f(1/10)$. Express your answer as a single simplified fraction.

$\frac{1}{10}$ is near $a=0$.

$f(x) = \dfrac{1}{\sqrt{1+x}}$; $f(0) = \dfrac{1}{\sqrt{1+0}} = 1$

$f'(x) = \dfrac{d}{dx}\left[(1+x)^{-\frac{1}{2}}\right] = -\frac{1}{2}(1+x)^{-3/2}$; $f'(0) = -\frac{1}{2}$

$T_1(x) = f(a) + f'(a)(x-a) = 1 - \frac{1}{2}(x-0) = 1 - \frac{1}{2}x$,

$f\left(\frac{1}{10}\right) \approx T_1\left(\frac{1}{10}\right) = 1 - \frac{1}{2}\left(\frac{1}{10}\right) = \boxed{\dfrac{19}{20}}$

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(b) Give an error bound for your approximation in part (a). Again, express your answer as a single simplified fraction.

$R_1(x) = \dfrac{f''(z)}{2}(x-a)^2$ where $x = \frac{1}{10}$, $a=0$, $0<z<\frac{1}{10}$

$f'(z) = -\frac{1}{2}(1+z)^{-3/2}$, so $f''(z) = \frac{3}{4}(1+z)^{-5/2} = \dfrac{3}{4(1+z)^{5/2}}$.

$\therefore \left|R_1\left(\frac{1}{10}\right)\right| = \left|\left(\frac{1}{2}\right)\left(\dfrac{3}{4(1+z)^{5/2}}\right)\left(\frac{1}{10}-0\right)^2\right|$

$\leqslant \left|\left(\frac{1}{2}\right)\left(\dfrac{3}{4(1+0)^{5/2}}\right)\left(\frac{1}{10}\right)^2\right|$

$= \boxed{\dfrac{3}{800}}$

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Question 2:

(a) Find $T_2(x)$, Taylor polynomial of degree 2 for $f(x) = (x+2)e^{(x-1)}$ at $a = 1$.

$f(x) = (x+2)e^{x-1}$; $f(1) = (1+2)e^{1-1} = 3$

$f'(x) = e^{x-1} + (x+2)e^{x-1} = (x+3)e^{x-1}$; $f'(1) = (1+3)e^{1-1} = 4$

$f''(x) = e^{x-1} + (x+3)e^{x-1} = (x+4)e^{x-1}$; $f''(1) = (1+4)e^{1-1} = 5$

$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

$\boxed{T_2(x) = 3 + 4(x-1) + \frac{5}{2}(x-1)^2}$

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(b) Suppose $T_2(x)$ in part (a) is used to approximate $f(9/10)$. Give an error bound on the approximation. Express your answer as a single simplified fraction. (Note: you are not being asked to find the approximation to $f(9/10)$ here, but only the error bound associated with the approximation.)

$R_2(x) = \frac{f'''(z)}{3!}(x-a)^3$ where $x = \frac{9}{10}$, $a = 1$, $\frac{9}{10} < z < 1$.

$f'''(z) = (z+5)e^{z-1}$,

So $\left| R_2\left(\frac{9}{10}\right) \right| = \left| \left(\frac{1}{3!}\right)(z+5)e^{z-1}\left(\frac{9}{10}-1\right)^3 \right|$

$\leq \left| \cancel{\left(\frac{1}{3!}\right)}\cancel{(1+5)}\cancel{e^{1-1}}\,1 \left(-\frac{1}{10}\right)^3 \right|$

$= \boxed{\frac{1}{1000}}$

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Question 3:
Find the Taylor series about $a = -1$ for $f(x) = 1 + 4x + 3x^2 + 2x^3$. You should be able to write all terms of the series.

$f(x) = 1 + 4x + 3x^2 + 2x^3$; $f(-1) = 1 + 4(-1) + 3(-1)^2 + 2(-1)^3 = -2$

$f'(x) = 4 + 6x + 6x^2$; $f'(-1) = 4 + 6(-1) + 6(-1)^2 = 4$

$f''(x) = 6 + 12x$; $f''(-1) = 6 + 12(-1) = -6$

$f'''(x) = 12$; $f'''(-1) = 12$

$f^{(k)}(x) = 0$ for $k \geq 3$.

$$T(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$T(x) = -2 + 4(x+1) - \frac{6}{2}(x+1)^2 + \frac{12}{3!}(x+1)^3$$

$$= \boxed{-2 + 4(x+1) - 3(x+1)^2 + 2(x+1)^3}$$

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Question 4: Find the first four nonzero terms of the Taylor series about $a = -2$ for $g(x) = \dfrac{5}{3-2x}$ and state the open interval of convergence. (Hint: think about the Maclaurin series for $\dfrac{1}{1-x}$.)

$$g(x) = \frac{5}{3-2x} = \frac{5}{3-2(x+2)+4} = \frac{5}{7-2(x+2)} = \frac{5}{7}\left[\frac{1}{1-\frac{2}{7}(x+2)}\right].$$

$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \cdots$ for $|u| < 1$.

Letting $u = \frac{2}{7}(x+2)$:

$$g(x) = \frac{5}{7}\left[1 + \frac{2}{7}(x+2) + \left(\frac{2}{7}(x+2)\right)^2 + \left(\frac{2}{7}(x+2)\right)^3 + \cdots\right]$$

$$= \boxed{\frac{5}{7} + \frac{10}{7^2}(x+2) + \frac{20}{7^3}(x+2)^2 + \frac{40}{7^4}(x+2)^3 + \cdots}$$

Series is valid for

$\left|\frac{2}{7}(x+2)\right| < 1 \Rightarrow |x-(-2)| < \frac{7}{2}$

$-7/2$ $\quad$ $7/2$

$-\frac{11}{2}$ $\quad$ -2 $\quad$ $3/2$

$\boxed{\therefore I = \left(-\frac{11}{2}, \frac{3}{2}\right)}$

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Question 5: Find the Maclaurin polynomial of degree 11 for $f(x) = x^2 \arctan(2x^3)$.

$$\arctan(u) = u - \frac{u^3}{3} + \frac{u^5}{5} - \frac{u^7}{7} + \cdots$$

Letting $u = 2x^3$:

$$\arctan(2x^3) = 2x^3 - \frac{(2x^3)^3}{3} + \frac{(2x^3)^5}{5} - \frac{(2x^3)^7}{7} + \cdots$$

$$\therefore \quad x^2\arctan(2x^3) = x^2\left[2x^3 - \frac{(2x^3)^3}{3} + \frac{(2x^3)^5}{5} - \frac{(2x^3)^7}{7} + \cdots\right]$$

$$= \underbrace{2x^5 - \frac{8x^{11}}{3}}_{T_{11}(x)} + \frac{32x^{17}}{5} - \frac{128x^{23}}{7} + \cdots$$

$$\boxed{\therefore \quad T_{11}(x) = 2x^5 - \frac{8x^{11}}{3}}$$

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Question 6: Use series (and not L'Hospital's Rule) to find the limit: $\lim_{x\to 0} \dfrac{e^{x^3} - 1 - x^3}{\sin(x^3) \bullet x^3}$

$$e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{3!} + \cdots$$

let $u = x^3$:

$$e^{x^3} = 1 + x^3 + \frac{x^6}{2} + \frac{x^9}{3!} + \cdots$$

$$\sin(u) = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots$$

let $u = x^3$:

$$\sin(x^3) = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots$$

so $$x^3\sin(x^3) = x^6 - \frac{x^{12}}{3!} + \frac{x^{18}}{5!} - \cdots$$

$$\lim_{x\to 0} \frac{e^{x^3} - 1 - x^3}{x^3\sin(x^3)}$$

$$= \lim_{x\to 0} \frac{(\cancel{1} + \cancel{x^3} + \frac{x^6}{2} + \frac{x^9}{3!} + \cdots) \cancel{-1} \cancel{-x^3}}{x^6 - \frac{x^{12}}{3!} + \frac{x^{18}}{5!} - \cdots}$$

$$= \lim_{x\to 0} \frac{\cancel{x^6}\left(\frac{1}{2} + \frac{x^3}{3!} + \cdots\right)}{\cancel{x^6}\left(1 - \frac{x^6}{3!} + \frac{x^{12}}{5!} - \cdots\right)}$$

$$= \boxed{\frac{1}{2}}$$

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Question 7: Find the first three non-zero terms of the Maclaurin series for $f(x) = e^{-x}\cos(x)$.

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$\cos(x) \backslash e^{-x}$	1	$-x$	$+\frac{x^2}{2}$	$-\frac{x^3}{3!}$	$\cdots$
1	1	$-x$	$+\frac{x^2}{2}$	$-\frac{x^3}{3!}$	
$-\frac{x^2}{2}$	$-\frac{x^2}{2}$	$+\frac{x^3}{2}$	$-\frac{x^4}{4}$	$+\frac{x^5}{12}$	
$\frac{x^4}{4!}$	$\frac{x^4}{4!}$	$-\frac{x^5}{4!}$	$+\frac{x^6}{48}$	$-\frac{x^7}{144}$	

$$e^{-x}\cos(x)$$

$$= 1 - x + \underbrace{\left(\cancel{\frac{x^2}{2} - \frac{x^2}{2}}\right)}_{\to 0} + \left(-\frac{x^3}{3!} + \frac{x^3}{2}\right) + \cdots$$

$$= \boxed{1 - x + \frac{x^3}{3} - \cdots}$$

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Question 8: Find the radius of convergence R and open interval of convergence $\mathcal{I}$ for the power series

$$f(x) = \sum_{k=1}^{\infty} \frac{2^k (x+1)^{2k}}{k^2} \quad \Big\} \quad a = -1, \; u_k(x) = \frac{2^k (x+1)^{2k}}{k^2}$$

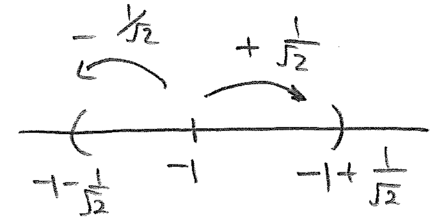
$$\lim_{k\to\infty} \left| \frac{u_{k+1}(x)}{u_k(x)} \right| < 1$$

$$\lim_{k\to\infty} \left| \frac{2^{k+1}(x+1)^{2(k+1)}}{(k+1)^2} \cdot \frac{k^2}{2^k (x+1)^{2k}} \right| < 1$$

$$\lim_{k\to\infty} \underbrace{\left| \frac{2^{k+1}}{2^k} \right|}_{=2} \cdot \underbrace{\left| \frac{k^2}{(k+1)^2} \right|}_{\to 1} \cdot \underbrace{\left| \frac{(x+1)^{2k+2}}{(x+1)^{2k}} \right|}_{= |x+1|^2} < 1$$

so $\quad 2|x+1|^2 < 1$

$$|x - (-1)| < \frac{1}{\sqrt{2}}$$



$$\therefore \; R = \frac{1}{\sqrt{2}}$$

$$\mathcal{I} = \left(-1 - \frac{1}{\sqrt{2}}, -1 + \frac{1}{\sqrt{2}}\right)$$

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