Question 1: Determine $\int x^2 e^{-2x}\,dx = I$

$u = x^2 \quad dv = e^{-2x}dx$

$du = 2x\,dx \quad v = -\frac{e^{-2x}}{2}$

$$I = \int u\,dv = uv - \int v\,du$$

$$= -\frac{x^2e^{-2x}}{2} - \int -\frac{e^{-2x}}{\cancel{2}}\cdot\cancel{2}x\,dx$$

$$= -\frac{x^2e^{-2x}}{2} + \underbrace{\int xe^{-2x}\,dx}$$

$u = x, \quad dv = e^{-2x}dx$

$du = dx, \quad v = -\frac{e^{-2x}}{2}$

$$= -\frac{x^2e^{-2x}}{2} + (x)\left(-\frac{e^{-2x}}{2}\right) - \int -\frac{e^{-2x}}{2}\,dx$$

$$= -\frac{x^2e^{-2x}}{2} - \frac{xe^{-2x}}{2} + \frac{1}{2}\int e^{-2x}\,dx$$

$$= \boxed{-\frac{x^2e^{-2x}}{2} - \frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} + C}$$

[5]

Question 2: Determine $\int \tan^3(x)\sec^3(x)\,dx$

$$= \int \tan^2(x)\sec^2(x)\,\sec(x)\tan(x)\,dx$$

$$= \int (\sec^2(x)-1)\sec^2(x)\,\sec(x)\tan(x)\,dx$$

$u = \sec(x)$

$du = \sec(x)\tan(x)dx$

$$= \int (u^2-1)u^2\,du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \boxed{\frac{\sec^5(x)}{5} - \frac{\sec^3(x)}{3} + C}$$

[5]

Question 3: Determine $\int \frac{1}{x^2\sqrt{16+x^2}}\,dx = I$

Let $x = 4\tan\theta$, $dx = 4\sec^2\theta\,d\theta$ $\Big\}$ $\therefore \tan\theta = \frac{x}{4}$

$$\therefore I = \int \frac{1 \cdot 4\sec^2\theta}{16\tan^2\theta\sqrt{16+16\tan^2\theta}}\,d\theta$$

$$= \int \frac{\cancel{4}\sec^{\cancel{2}}\theta}{16\tan^2\theta \cdot \cancel{4} \cdot \cancel{\sec\theta}}\,d\theta$$

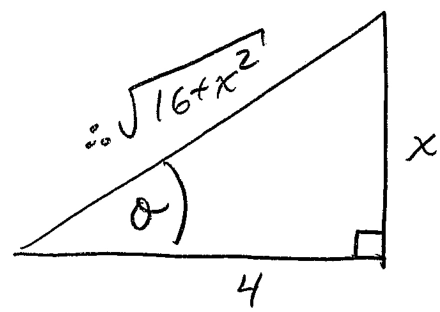
$$= \frac{1}{16}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta \quad \Big\{ \begin{array}{l} u = \sin\theta \\ du = \cos\theta\,d\theta \end{array}$$

$$= \frac{1}{16}\int \frac{1}{u^2}\,du$$

$$= -\frac{1}{16}\cdot\frac{1}{u} + C$$

$$= -\frac{1}{16}\,\frac{1}{\sin\theta} + C$$

$$= \boxed{-\frac{1}{16}\cdot\frac{\sqrt{16+x^2}}{x} + C}$$



[10]

Question 4: Determine $\displaystyle\int \frac{13x}{(x+2)(x^2+9)}\,dx$

$$\frac{13x}{(x+2)(x^2+9)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+9}$$

$$= \frac{A(x^2+9)+(Bx+C)(x+2)}{(x+2)(x^2+9)}$$

$$= \frac{(A+B)x^2+(2B+C)x+(9A+2C)}{(x+2)(x^2+9)}$$

$\therefore$ $A+B=0 \Rightarrow B=-A$

$2B+C=13 \Rightarrow C=13-2B= 13+2A$

$9A+2C=0 \Rightarrow 9A+2(13+2A)=0 \Rightarrow 13A=-26 \Rightarrow A=-2$

$\therefore B=-(-2)=2$

$C=13+2(-2)=9$

$$\therefore\ I = \int \frac{-2}{x+2} + \frac{2x+9}{x^2+9}\,dx$$

$$= -2\int \frac{1}{x+2}\,dx + \underbrace{\int \frac{2x}{x^2+9}\,dx}_{\substack{u=x^2+9 \\ du=2x\,dx}} + 9\underbrace{\int \frac{1}{x^2+9}\,dx}_{\text{Formula \#18}}$$

$$= -2\ln|x+2| + \ln|x^2+9| + \frac{\cancel{9}^{3}}{\cancel{3}}\arctan\left(\frac{x}{3}\right) + C$$

$$= \boxed{-2\ln|x+2| + \ln|x^2+9| + 3\arctan\left(\frac{x}{3}\right) + C}$$

[10]

Question 5: For this question consider the integral

$$\int_0^{2\pi} 2^{(\sin^2 x - 1)}\, dx$$

(i) Determine T_4, the trapezoid rule on four subintervals. Simplify your answer to a single numerical value.

x:	0	$\pi/2$	π	$3\pi/2$	2π
$f(x)$:	$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{1}{2}$

$\Delta x = \pi/2$

$$T_4 = \frac{\Delta x}{2}\left[f(0) + 2f\left(\tfrac{\pi}{2}\right) + 2f(\pi) + 2f\left(\tfrac{3\pi}{2}\right) + f(2\pi)\right]$$

$$= \frac{(\pi/2)}{2}\left[\tfrac{1}{2} + (2)(1) + (2)\left(\tfrac{1}{2}\right) + (2)(1) + \tfrac{1}{2}\right]$$

$$= \boxed{\frac{3\pi}{2}}$$

[5]

(ii) It can be shown that for the integrand $f(x) = 2^{(\sin^2 x - 1)}$, above,

$$-\frac{3}{2} \leq f''(x) \leq 1$$

So $|f''(x)| \leq \frac{3}{2}$

for every x in $[0, 2\pi]$.

Use this information to determine an error bound on your approximation in part (i). State your answer as a single simplified fraction.

$$|E_{T_4}| \leq \frac{K(b-a)^3}{12n^2}$$ where $K = \frac{3}{2}$, $[a,b] = [0, 2\pi]$, $n = 4$

$$= \frac{3/2\,(2\pi - 0)^3}{12 \cdot 4^2}$$

$$= \frac{\cancel{3}}{2} \cdot \frac{\cancel{2^3}\pi^3}{\cancel{12} \cdot \cancel{4^2}}$$

$$= \boxed{\frac{\pi^3}{16}}$$

[5]

Question 6: Determine if the improper integral $\int_1^2 \frac{x}{\sqrt{x^2-1}}\,dx$ converges or diverges. If it converges give the value, if it diverges then say so.

$$\int_1^2 \frac{x}{\sqrt{x^2-1}}\,dx = \lim_{a\to 1^+} \int_a^2 \frac{x}{\sqrt{x^2-1}}\,dx \quad \left.\begin{matrix} u = x^2-1 \\ du = 2x\,dx \end{matrix}\right\} \quad \begin{matrix} x=a \Rightarrow u = a^2-1 \\ x=2 \Rightarrow u = 3 \end{matrix}$$

$$= \lim_{a\to 1^+} \frac{1}{2}\int_{a^2-1}^{3} u^{-1/2}\,du$$

$$= \lim_{a\to 1^+} \left[u^{1/2}\right]_{a^2-1}^{3}$$

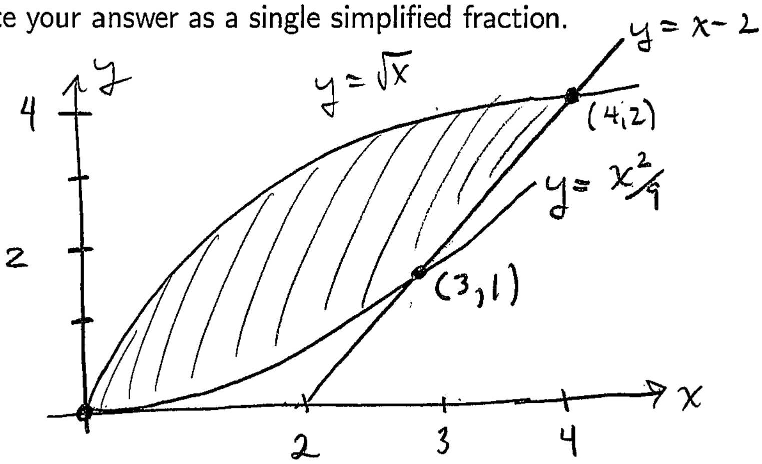
$$= \lim_{a\to 1^+} 3^{1/2} - \underbrace{(a^2-1)^{1/2}}_{\to 0}$$

$$= \sqrt{3}$$

$\boxed{\therefore \text{ Integral converges to } \sqrt{3}}$

[5]

Question 7: Determine the area of the region lying below $y = \sqrt{x}$, above $y = x^2/9$ and to the left of $y = x-2$. State your answer as a single simplified fraction.



$$\therefore A = \int_0^3 x^{1/2} - \frac{x^2}{9}\,dx + \int_3^4 x^{1/2} - (x-2)\,dx$$

$$= \frac{2}{3}\left[x^{3/2}\right]_0^3 - \frac{1}{27}\left[x^3\right]_0^3 + \frac{2}{3}\left[x^{3/2}\right]_3^4 - \frac{1}{2}\left[x^2\right]_3^4 + \left[2x\right]_3^4$$

$$= \cancel{\frac{2}{3}(3^{3/2})} - \frac{1}{27}(27) + \frac{2}{3}(4^{3/2} - \cancel{3^{3/2}}) - \frac{1}{2}[16-9] + [8-6]$$

$$= -1 + \frac{16}{3} - \frac{7}{2} + 2$$

$$= \boxed{\frac{17}{6}}$$

[5]