Question 1:

(a) A car travelling at 30 m/s applies the brakes and accelerates at $a(t) = -10t - 5$ m/s^2, where $t = 0$ corresponds to the instant the brakes are applied (notice the acceleration is negative since the car is slowing down.) How long does it take the car to come to a stop?

$$v'(t) = -10t - 5, \quad v(0) = 30$$

$$\therefore \quad v(t) = -\frac{10t^2}{2} - 5t + C$$

$$= -5t^2 - 5t + C$$

$v(0) = 30$, so $30 = -5 \cdot 0^2 - 5 \cdot 0 + C \Rightarrow C = 30$

$$\therefore \quad v(t) = -5t^2 - 5t + 30$$

At rest $v = 0$, so $-5t^2 - 5t + 30 = 0$

$$-5(t^2 + t - 6) = 0$$

$$-5(t+3)(t-2) = 0$$

$$\therefore \quad \cancel{t = -3} \text{ or } \boxed{t = 2s}$$

[5]

(b) Find $f(x)$ if

$$f''(x) = 28\sqrt[3]{x} + \sin(x), \quad f'(0) = 1, \quad f(0) = \pi$$

$$f''(x) = 28x^{1/3} + \sin(x)$$

$$f'(x) = 28 \cdot \frac{3}{4} x^{4/3} - \cos(x) + C_1$$

$$= 21x^{4/3} - \cos(x) + C_1$$

$$f'(0) = 1 \Rightarrow 1 = 21 \cdot 0^{4/3} - \cos(0) + C_1$$

$$1 = -(+1) + C_1$$

$$\therefore C_1 = 2$$

$$f'(x) = 21x^{4/3} - \cos(x) + 2$$

$$f(x) = \overset{3}{\cancel{21}} \cdot \frac{3}{\cancel{7}} x^{7/3} - \sin(x) + 2x + C_2$$

$$f(0) = \pi \Rightarrow 0 - \sin(0) + 0 + C_2 = \pi$$

$$\therefore C_2 = \pi$$

$$\boxed{\therefore f(x) = 9x^{7/3} - \sin(x) + 2x + \pi}$$

[5]

Question 2: Determine the following:

(a) $\int x(x^2-3)\,dx = \int x^3 - 3x\,dx$

$$= \boxed{\frac{x^4}{4} - \frac{3x^2}{2} + C}$$

[2]

(b) $\int \frac{x^2+2\sqrt{x}+1}{x}\,dx = \int x + 2x^{-1/2} + \frac{1}{x}\,dx$

$$= \boxed{\frac{x^2}{2} + 4 \cdot x^{1/2} + \ln|x| + C}$$

[2]

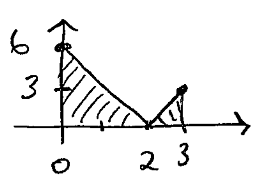
(c) $\int_0^{\pi} 2\sin(x) - \frac{\cos(x)}{3}\,dx = -2\Big[\cos(x)\Big]_0^{\pi} - \frac{1}{3}\Big[\sin(x)\Big]_0^{\pi}$

$$= -2\Big[\underbrace{\cos(\pi)}_{-1} - \underbrace{\cos(0)}_{1}\Big] - \frac{1}{3}\Big[\underbrace{\sin(\pi)}_{0} - \underbrace{\sin(0)}_{0}\Big]$$

$$= \boxed{4}$$

[3]

(d) $\int_0^3 |3x-6|\,dx$ $\quad y = |3x-6|$ has graph



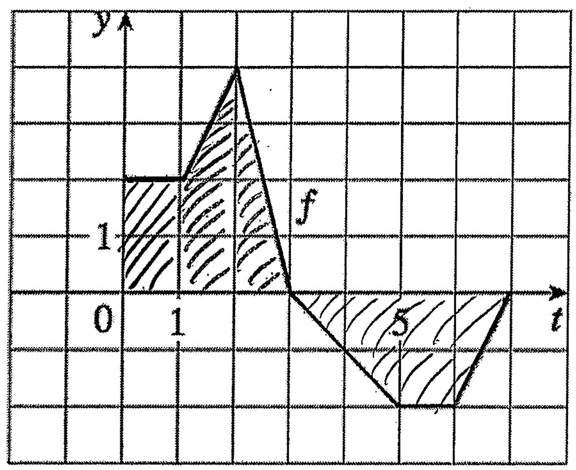
$$\therefore \int_0^3 |3x-6|\,dx = \left(\tfrac{1}{2}\right)(2)(6) + \left(\tfrac{1}{2}\right)(1)(3)$$

$$= 6 + \frac{3}{2}$$

$$= \boxed{\frac{15}{2}}$$

[3]

Question 3: The graph of $y = f(t)$ below represents the outside temperature over a seven hour period of a January day. What was the average temperature over the seven hours?



$$\int_0^7 f(t)\,dt = 2+3+2-\frac{1}{2}-\frac{3}{2}-2-1 = 2$$

$$\therefore\ f_{ave} = \frac{1}{7-0}\int_0^7 f(t)\,dt = \left(\frac{1}{7}\right)(2) = \boxed{\frac{2}{7}}$$

[5]

Question 4: An ant colony of size 100 grows over time according at a rate given by $\dfrac{100t^2}{1+t^3}$ ants per week, where $t = 0$ corresponds to the present. What will be the ant colony size in two weeks time?

$$P'(t) = \frac{100t^2}{1+t^3}, \quad P(0) = 100.$$

$$\therefore\ P(2) - P(0) = \int_0^2 P'(t)\,dt$$

$$\Rightarrow P(2) = 100 + \int_0^2 \frac{100t^2}{1+t^3}\,dt \quad \left\}\begin{array}{ll} u = 1+t^3 & t=0 \Rightarrow u=1 \\ du = 3t^2\,dt & t=2 \Rightarrow u=9 \end{array}\right.$$

$$= 100 + \frac{100}{3}\int_1^9 \frac{1}{u}\,du$$

$$= 100 + \frac{100}{3}\big[\ln|u|\big]_1^9 = 100 + \frac{100}{3}\left(\ln|9| - \cancel{\ln|1|}^{\,0}\right) = \boxed{\frac{300+100\ln 9}{3}}$$

[5]

Question 5: Determine the following:

(a) $\int x(1-x^2)^{1/2}\,dx = I$

let $u = 1-x^2$
$du = -2x\,dx$

$\therefore I = -\frac{1}{2}\int u^{1/2}\,du$

$= -\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}\cdot u^{3/2} + C$

$= \boxed{-\frac{1}{3}(1-x^2)^{3/2} + C}$

[2]

(b) $\int \sec^2(3x+2)\,dx$

$= \boxed{\frac{1}{3}\tan(3x+2) + C}$

[2]

(c) $\int \frac{e^x}{(5-3e^x)}\,dx = I$

let $u = 5-3e^x$
$du = -3e^x\,dx$

$\therefore I = -\frac{1}{3}\int \frac{1}{u}\,du$

$= -\frac{1}{3}\ln|u| + C$

$= \boxed{-\frac{1}{3}\ln|5-3e^x| + C}$

[3]

(d) $\int x^3\sqrt{x^2+1}\,dx$ (Hint: $x^3 = x\cdot x^2$)

$I = \int x^2\sqrt{x^2+1}\,x\,dx$

let $u = x^2+1 \iff x^2 = u-1$
$du = 2x\,dx$

$\therefore I = \frac{1}{2}\int (u-1)u^{1/2}\,du$

$= \frac{1}{2}\int u^{3/2} - u^{1/2}\,du$

$= \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{5}u^{5/2} - \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}u^{3/2} + C$

$= \boxed{\frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C}$

[3]

Question 6: Determine the following:

(a) $\displaystyle\int_0^1 \frac{10\sqrt{x}}{(1+x^{3/2})^2}\,dx = I$

$u = 1 + x^{3/2}$, $du = \frac{3}{2}\cdot x^{1/2}$; $x=0 \Rightarrow u=1$, $x=1 \Rightarrow u=2$

$\therefore I = 10\cdot\frac{2}{3}\int_1^2 u^{-2}\,du$

$= \frac{20}{3}\left[\frac{-1}{u}\right]_1^2$

$= \frac{20}{3}\left[\frac{-1}{2}+1\right]$

$= \left(\frac{20}{3}\right)\left(\frac{+1}{2}\right)$

$= \boxed{\frac{+10}{3}}$

[3]

(b) $\displaystyle\int_0^{\pi} \frac{\sin(t)}{2-\cos(t)}\,dt = I$

let $u = 2-\cos(t)$, $du = \sin(t)$; $t=0 \Rightarrow u=1$, $t=\pi \Rightarrow u=3$

$\therefore I = \int_1^3 \frac{1}{u}\,du$

$= \left[\ln|u|\right]_1^3$

$= \ln|3| - \cancel{\ln|1|}^{\,0}$

$= \boxed{\ln 3}$

[3]

(c) $\displaystyle\int_e^{e^2} \frac{1}{x[\ln(x)]^3}\,dx = I$

let $u = \ln(x)$, $du = \frac{1}{x}\,dx$; $x=e \Rightarrow u=\ln(e)=1$, $x=e^2 \Rightarrow u=\ln(e^2)=2$

$\therefore I = \int_1^2 \frac{1}{u^3}\,du$

$= -\frac{1}{2}\left[\frac{1}{u^2}\right]_1^2$

$= \frac{-1}{2}\left[\frac{1}{4}-1\right]$

$= \left(\frac{-1}{2}\right)\left(\frac{-3}{4}\right)$

$= \boxed{\frac{3}{8}}$

[4]