Question 1:

(a) Use a linear approximation $T_1(x)$ to estimate $f(-1/6)$ where $f(x) = (2+e^x)^2$. Simplify your final answer.

Here $a=0$.

$f(x) = (2+e^x)^2$; $f(0) = (2+e^0)^2 = 9$

$f'(x) = 2(2+e^x)\, e^x$; $f'(0) = 2(2+e^0)\, e^0 = 6$

$\therefore T_1(x) = f(a) + f'(a)(x-a)$

$T_1(x) = 9 + 6x$

$\therefore f(-\frac{1}{6}) \approx T_1(-\frac{1}{6}) = 9 + 6(-\frac{1}{6}) = 9 - 1 = \boxed{8}$

[5]

(b) Give an error bound on your approximation in part (a). Again, simplify your final answer.

$f'(z) = 2(2+e^z)e^z = 4e^z + 2e^{2z}$

$f''(z) = 4e^z + 4e^{2z}$.

$R_1(x) = \frac{1}{2} f''(z)\, x^2$

$|R_1(-\frac{1}{6})| = \left|\frac{1}{2}(4e^z + 4e^{2z})(-\frac{1}{6})^2\right|$, $-\frac{1}{6} < z < 0$

$\leq \left|\frac{1}{2}(4e^0 + 4e^{2\cdot 0})(\frac{-1}{6})^2\right|$

$= (\frac{1}{2})(8)(\frac{1}{36})$

$= \boxed{\frac{1}{9}}$

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Question 2:

(a) Use a Taylor polynomial of degree 2 about $a = 4$ to estimate $\sqrt{5}$. Simplify your final answer.

$f(x) = x^{1/2}$; $f(a) = f(4) = 2$

$f'(x) = \frac{1}{2}x^{-1/2}$; $f'(a) = f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

$f''(x) = -\frac{1}{4}x^{-3/2}$; $f''(a) = f''(4) = \frac{-1}{4(\sqrt{4})^3} = \frac{-1}{32}$

$\therefore T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

$= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$.

$\therefore \sqrt{5} \approx T_2(5) = 2 + \frac{1}{4}(5-4) - \frac{1}{64}(5-4)^2$

$= 2 + \frac{1}{4} - \frac{1}{64}$

$= \frac{128+16-1}{64} = \boxed{\frac{143}{64}}$

[5]

(b) Give an error bound on your approximation in part (a). Again, simplify your final answer.

$f'''(z) = \frac{3}{8} z^{-5/2}$.

$R_2(x) = \frac{f'''(z)}{3!}(x-a)^3$

$\therefore |R_2(5)| = \left| \frac{1}{3!} \cdot \frac{3}{8 z^{5/2}} \cdot (5-4)^3 \right|$, $4 < z < 5$

$\leq \left| \frac{1}{\cancel{3} \cdot 2} \cdot \frac{\cancel{3}}{2^3 \cdot (\sqrt{4})^5} \cdot 1^3 \right|$

$= \boxed{\frac{1}{2^9}}$ or $\frac{1}{512}$.

[5]

Question 3:

(a) Find the first three nonzero terms of the Maclaurin series for $f(x) = \dfrac{\arctan(2x)}{1+x^2}$. $= \arctan(2x) \cdot \dfrac{1}{1+x^2}$

$\arctan(u) = u - \dfrac{u^3}{3} + \dfrac{u^5}{5} - \cdots$

Let $u = 2x$: $\arctan(2x) = 2x - \dfrac{8}{3}x^3 + \dfrac{32}{5}x^5 - \cdots$

$\dfrac{1}{1-u} = 1 + u + u^2 + u^3 + \cdots$

Let $u = -x^2$: $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$

	$2x - \frac{8}{3}x^3 + \frac{32}{5}x^5 - \cdots$
$+1$	$2x - \frac{8}{3}x^3 + \frac{32}{5}x^5 - \cdots$
$-x^2$	$-2x^3 + \frac{8}{3}x^5 - \frac{32}{5}x^7 + \cdots$
$+x^4$	$2x^5 - \frac{8}{3}x^7 + \frac{32}{5}x^9 - \cdots$
$-x^6$	$-2x^7 + \frac{8}{3}x^9 - \frac{32}{5}x^{11} + \cdots$

$$\therefore \frac{\arctan(2x)}{1+x^2} = 2x + \left(-\frac{8}{3} - 2\right)x^3 + \left(2 + \frac{8}{3} + \frac{32}{5}\right)x^5 + \cdots$$

$$= \boxed{2x - \frac{14}{3}x^3 + \frac{166}{15}x^5 + \cdots}$$

[5]

(b) Find the first four nonzero terms of the Taylor series about $a = 2$ for $g(x) = \dfrac{2}{7-3x}$ and state the open interval of convergence.

$\dfrac{2}{7-3x}$

$= \dfrac{2}{7 - 3(x-2) - 6}$

$= \dfrac{2}{1 - [3(x-2)]}$

$= 2\left(\dfrac{1}{1-[3(x-2)]}\right)$

$= 2\left(1 + [3(x-2)] + [3(x-2)]^2 + [3(x-2)]^3 + \cdots\right)$

$= \boxed{2 + 6(x-2) + 18(x-2)^2 + 54(x-2)^3 + \cdots}$

Valid for $-1 < 3(x-2) < 1$

$\Rightarrow -\dfrac{1}{3} < x-2 < \dfrac{1}{3}$

$\Rightarrow \dfrac{5}{3} < x < \dfrac{7}{3}$

$\boxed{\therefore I = \left(\dfrac{5}{3}, \dfrac{7}{3}\right)}$

[5]

Question 4: The first three terms of the Maclaurin series for $\tan(x)$ is

$$\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

Use this to find the first three nonzero terms of the Maclaurin series for $g(x) = x^3 \sec^2(x)$. (There are several ways to do this, but one way is much easier than the others.)

$$x^3 \sec^2(x) = x^3 \cdot \frac{d}{dx}\left[\tan(x)\right]$$

$$= x^3 \cdot \frac{d}{dx}\left[x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots\right]$$

$$= x^3\left[1 + x^2 + \frac{2}{3}x^4 + \cdots\right]$$

$$= \boxed{x^3 + x^5 + \frac{2}{3}x^7 + \cdots}$$

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Question 5: Evaluate the following limit:

$$\lim_{x \to 0} \frac{\ln(1 - x^2) - e^{(-x^2)} + 1}{3x^4}$$

$$= \lim_{x \to 0} \frac{\left(-x^2 - \frac{x^4}{2} - \frac{x^6}{3} - \cdots\right) - \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \cdots\right) + 1}{3x^4}$$

$$= \lim_{x \to 0} \frac{\cancel{-x^2} - \frac{x^4}{2} - \frac{x^6}{3} - \cdots \cancel{-1} + \cancel{x^2} - \frac{x^4}{2} + \frac{x^6}{3!} - \cdots \cancel{+1}}{3x^4}$$

$$= \lim_{x \to 0} \frac{-x^4 - \frac{x^6}{3!} + \cdots}{3x^4}$$

$$= \lim_{x \to 0} \frac{\cancel{x^4}\left(-1 - \frac{x^2}{3!} + \cdots\right)}{3\cancel{x^4}} = \boxed{\frac{-1}{3}}$$

[6]

Question 6: Find the radius of convergence R and open interval of convergence $\mathcal{I}$ for the power series

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x+3)^{2k}}{9^k} \quad \Big\} \; a = -3$$

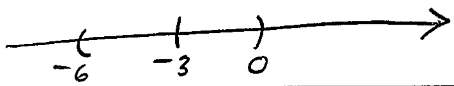
$$\lim_{k\to\infty} \left| \frac{u_{k+1}(x)}{u_k(x)} \right| < 1$$

$$\Rightarrow \lim_{k\to\infty} \left| \frac{(-1)^{k+1}}{(-1)^k} \cdot \frac{9^k}{9^{k+1}} \cdot \frac{(x+3)^{2(k+1)}}{(x+3)^{2k}} \right| < 1$$

$$\Rightarrow \lim_{k\to\infty} \frac{1}{9} |x+3|^2 < 1$$

$$\Rightarrow |x+3|^2 < 9$$

$$\Rightarrow |x+3| < 3$$



$$\boxed{\therefore\ R = 3, \quad \mathcal{I} = (-6, 0)}$$

[5]

Question 7: Find the radius of convergence R and open interval of convergence $\mathcal{I}$ for the power series

$$f(x) = \sum_{k=0}^{\infty} \frac{2^k \sqrt{k}\, x^k}{k!} \quad \Big\} \; a = 0$$

$$\lim_{k\to\infty} \left| \frac{u_{k+1}(x)}{u_k(x)} \right| < 1$$

$$\Rightarrow \lim_{k\to\infty} \left| \frac{2^{k+1}}{2^k} \cdot \frac{\sqrt{k+1}}{\sqrt{k}} \cdot \frac{k!}{(k+1)!} \cdot \frac{x^{k+1}}{x^k} \right| < 1$$

$$\Rightarrow \lim_{k\to\infty} \left| 2 \cdot \underbrace{\sqrt{\frac{k+1}{k}}}_{\to 1} \cdot \underbrace{\frac{1}{k+1}}_{\to 0} \cdot x \right| < 1$$

$$\Rightarrow 0 < 1 \quad \Big\}$$ true for all real x.

$$\boxed{\therefore\ R = \infty, \quad \mathcal{I} = (-\infty, \infty)}$$

[5]