Question 1: Determine $\int_0^{\pi/4} x\sec^2(x)\,dx = I$

$u = x \quad dv = \sec^2(x)dx$

$du = dx \quad v = \tan(x)$

$\therefore I = \int_0^{\pi/4} u\,dv$

$= [uv]_0^{\pi/4} - \int_0^{\pi/4} v\,du$

$= [x\tan(x)]_0^{\pi/4} - \int_0^{\frac{\pi}{4}} \tan(x)dx$

$= \frac{\pi}{4}\cdot 1 - 0 - \underbrace{\int_0^{\frac{\pi}{4}} \frac{\sin(x)}{\cos(x)}dx}_{\text{let } w=\cos(x),\ dw = -\sin(x)dx}$

$= \frac{\pi}{4} + [\ln|\cos(x)|]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{4} + \ln|\frac{1}{\sqrt{2}}| - \cancel{\ln|1|}^{\to 0} = \boxed{\frac{\pi}{4} + \ln(\frac{1}{\sqrt{2}})}$

[5]

Question 2: Determine $\int \tan^6(x)\sec^4(x)\,dx$

$= \int \tan^6(x)\sec^2(x)\sec^2(x)dx$

$= \int \tan^6(x)(1+\tan^2(x))\sec^2(x)dx$ $\quad \left\{ \begin{array}{l} u = \tan(x) \\ du = \sec^2(x)dx \end{array}\right.$

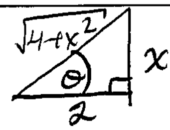
$= \int u^6(1+u^2)\,du$

$= \frac{u^7}{7} + \frac{u^9}{9} + C$

$= \boxed{\frac{\tan^7(x)}{7} + \frac{\tan^9(x)}{9} + C}$

[5]

Question 3: Determine $I = \int \frac{\sqrt{4+x^2}}{x^4}\,dx$ $\left\}\begin{array}{l} x = 2\tan\theta \\ dx = 2\sec^2\theta\,d\theta \end{array}\right\}$



$$\therefore\ I = \int \frac{\sqrt{4+4\tan^2\theta}}{2^4\tan^4\theta}\,2\sec^2\theta\,d\theta$$

$$= \int \frac{\cancel{2}\sec\theta \cdot \cancel{2}\sec^2\theta}{\cancel{2^4}^{\,2^2}\tan^4\theta}\,d\theta$$

$$= \frac{1}{4}\int \frac{\cancel{\cos^4}\theta}{\sin^4\theta}\cdot\frac{1}{\cancel{\cos^3}\theta}\,d\theta \quad \left\}\begin{array}{l} \text{let } u = \sin\theta \\ du = \cos\theta\,d\theta \end{array}\right.$$

$$= \frac{1}{4}\int u^{-4}\,du$$

$$= \frac{-1}{12}\cdot\frac{1}{u^3} + C$$

$$= \frac{-1}{12}\cdot\frac{1}{\sin^3\theta} + C$$

$$= \frac{-1}{12}\cdot\frac{1}{\left(\frac{x}{\sqrt{4+x^2}}\right)^3} + C$$

$$= \boxed{-\frac{1}{12}\,\frac{(4+x^2)^{3/2}}{x^3} + C}$$

[5]

Question 4: Determine $\int \frac{x}{x^2+6x+10}\,dx$

$$= \frac{1}{2}\int \frac{2x}{x^2+6x+10}\,dx$$

$$= \frac{1}{2}\int \frac{2x+6\ \ -6}{x^2+6x+10}\,dx$$

$$= \underbrace{\frac{1}{2}\int \frac{2x+6}{x^2+6x+10}\,dx}_{\substack{u = x^2+6x+10 \\ du = (2x+6)\,dx}} - \underbrace{\frac{6}{2}\int \frac{1}{(x+3)^2+1}\,dx}_{\text{Formula \#18}}$$

$$= \boxed{\frac{1}{2}\ln|x^2+6x+10| - 3\cdot\arctan(x+3) + C}$$

[5]

Question 5: Determine $\int \frac{x+2}{x^2(x-1)}\,dx$

$$\frac{x+2}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} = \frac{Ax(x-1)+B(x-1)+Cx^2}{x^2(x-1)}$$

$$= \frac{(A+C)x^2 + (-A+B)x + (-B)}{x^2(x-1)}$$

$$\therefore \left.\begin{aligned} A+C &= 0 \quad (1) \\ -A+B &= 1 \quad (2) \\ -B &= 2 \quad (3) \end{aligned}\right\} \therefore \begin{aligned} B &= -2 \\ A &= B-1 = -3 \\ C &= -A = 3 \end{aligned}$$

$$\therefore I = \int -\frac{3}{x} - \frac{2}{x^2} + \frac{3}{x-1}\,dx$$

$$= \boxed{-3\ln|x| + \frac{2}{x} + 3\ln|x-1| + C}$$

[5]

Question 6: Evaluate $\int_0^\infty \frac{1}{(1+x^2)(1+\arctan(x))}\,dx$ making proper use of required limits.

First, for $I = \int \frac{1}{(1+x^2)(1+\arctan(x))}\,dx$, let $u = 1+\arctan(x)$, $du = \frac{1}{1+x^2}$

$$\therefore I = \int \frac{1}{u}\,du = \ln|u| = \ln|1+\arctan(x)| + C.$$

$$\therefore \int_0^\infty \frac{1}{(1+x^2)(1+\arctan(x))}\,dx$$

$$= \lim_{b\to\infty} \int_0^b \frac{1}{(1+x^2)(1+\arctan(x))}\,dx$$

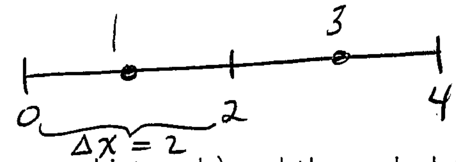
$$= \lim_{b\to\infty} \Big[\ln|1+\arctan(x)|\Big]_0^b$$

$$= \lim_{b\to\infty} \Big[\ln|1+\arctan(b)| - \underbrace{\cancel{\ln}|1+\arctan(0)|}_{\to 0}\Big] = \boxed{\ln\left|1+\frac{\pi}{2}\right|}$$

[5]

Question 7: For this question consider the integral

$$\int_0^4 \left(2x^2 - \frac{x^3}{2}\right) dx$$



(i) Determine both T_2 and M_2 (both trapezoid and midpoint rules on two subintervals) and then calculate $(1/3)T_2 + (2/3)M_2$.

$$T_2 = \frac{\Delta x}{2}\left[f(0) + 2f(2) + f(4)\right]$$

$$= \frac{2}{2}\left[0 + 2(8-4) + \cancel{(32-32)}\right]$$

$$= \boxed{8}$$

$$M_2 = \Delta x\left[f(1) + f(3)\right]$$

$$= 2\left[\left(2-\tfrac{1}{2}\right) + \left(18 - \tfrac{27}{2}\right)\right]$$

$$= \boxed{12}$$

$$\therefore \left(\tfrac{1}{3}\right)T_2 + \left(\tfrac{2}{3}\right)M_2 = \left(\tfrac{1}{3}\right)(8) + \left(\tfrac{2}{3}\right)(12) = \boxed{\frac{32}{3}}$$

[5]

(ii) How many subintervals are required to guarantee that the error in using the midpoint rule is at most 3/4?

(Recall: the error in using the midpoint rule to approximate $\int_a^b f(x)\,dx$ using n subintervals is at most $\frac{K(b-a)^3}{24n^2}$ where $|f''(x)| \le K$ on $[a, b]$.)

$$f(x) = 2x^2 - \frac{x^3}{2}$$

$$f'(x) = 4x - \frac{3x^2}{2}$$

$$f''(x) = 4 - 3x$$

$$|f''(x)| \le |f''(4)| = |4 - 3(4)| = 8 \leftarrow K.$$

We want

$$\frac{K(b-a)^3}{24n^2} \le \frac{3}{4}$$

$$\Rightarrow \frac{8(4-0)^3}{24n^2} \le \frac{3}{4}$$

$$\Rightarrow \frac{\cancel{8}\cdot 4^3}{\cancel{24}_3} \cdot \frac{4}{3} \le n^2$$

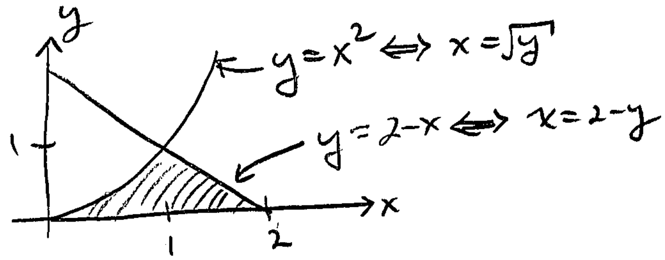
$$\Rightarrow \frac{4^4}{3^2} \le n^2$$

$$\Rightarrow \frac{4^2}{3} \le n$$

$$\therefore \boxed{n \ge 6}$$

[5]

Question 8: ~~Determine~~ Determine the area of the region in the first quadrant that is bounded by the curves $y = x^2$, $y = 2 - x$ and the x-axis.



Option 1: $A = \int_0^1 x^2 - 0 \, dx + \int_1^2 (2-x) - 0 \, dx$

Option 2: $A = \int_0^1 (2-y) - y^{1/2} \, dy$

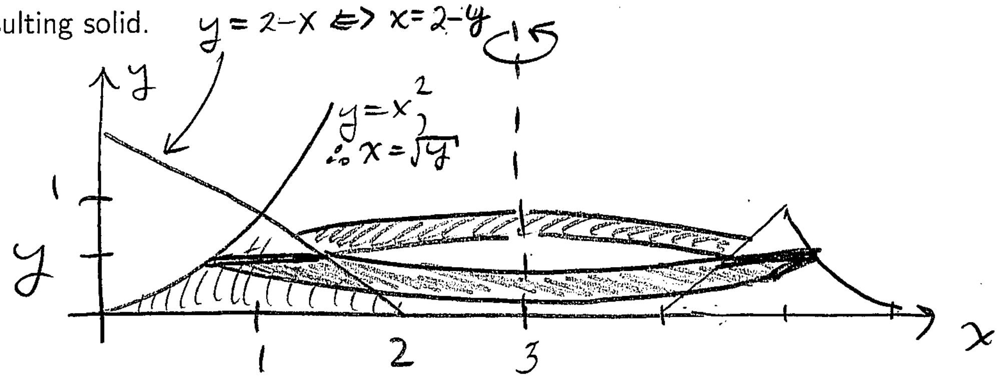
$= \left[2y - \frac{y^2}{2} - \frac{2}{3} y^{3/2} \right]_0^1$

$= 2 - \frac{1}{2} - \frac{2}{3}$

$= \boxed{\frac{5}{6}}$

[5]

Question 9: The region in the first quadrant that is bounded by the curves $y = x^2$, $y = 2 - x$ and the x-axis is rotated about the vertical line $x = 3$. Set up BUT DO NOT EVALUATE the integral representing the volume of the resulting solid.



$$V = \int_{y=0}^{1} \pi (3 - \sqrt{y})^2 - \pi (3 - (2-y))^2 \, dy$$

$$= \boxed{\int_0^1 \pi \left[(3 - y^{1/2})^2 - (1+y)^2 \right] dy}$$

[5]