Question 1: Find the radius of convergence R and open interval of convergence $\mathcal{I}$ for the power series

$$f(x) = \sum_{k=0}^{\infty} \frac{2^{2k}(x+3)^k}{k!}$$

$$\lim_{k\to\infty} \left| \frac{u_{k+1}(x)}{u_k(x)} \right| < 1$$

$$\Rightarrow \lim_{k\to\infty} \left| \frac{2^{2(k+1)}(x+3)^{k+1}}{(k+1)!} \cdot \frac{k!}{2^{2k}(x+3)^k} \right| < 1$$

$$\Rightarrow \lim_{k\to\infty} \left| \frac{2^{\cancel{2k}+2}}{2^{\cancel{2k}}} \cdot \frac{k!}{(k+1)!} \cdot \frac{(x+3)^{\cancel{k}+1}}{(x+3)^{\cancel{k}}} \right| < 1$$

$$\Rightarrow \lim_{k\to\infty} \left| 4 \cdot \frac{1}{k+1} \cdot (x+3) \right| < 1$$

$$\Rightarrow 0 < 1$$

$$\therefore \boxed{\mathcal{I} = (-\infty, \infty), \quad R = \infty.}$$

[5]

Question 2: Determine $f(x)$ if

$$f''(x) = 12x^2 - \sin(x), \quad f(0) = 1, \quad f'(0) = 0$$

$$f'(x) = \frac{\cancel{12}^{4}}{\cancel{3}} x^3 + \cos(x) + C_1$$

$$f'(0) = 0 \Rightarrow 4 \cdot 0^3 + \cos(0) + C_1 = 0$$

$$\Rightarrow 1 + C_1 = 0$$

$$\Rightarrow C_1 = -1$$

$$\therefore f'(x) = 4x^3 + \cos(x) - 1$$

$$\therefore f(x) = \frac{\cancel{4}x^4}{\cancel{4}} + \sin(x) - x + C_2$$

$$f(0) = 1 \Rightarrow 0^4 + \sin(0) - 0 + C_2 = 1$$

$$\Rightarrow C_2 = 1$$

$$\boxed{\therefore f(x) = x^4 + \sin(x) - x + 1}$$

[5]

Question 3: Use the definition of the definite integral in the form

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

to evaluate

$$\int_1^2 (3x^2 - 1)\,dx$$

Carefully set up the Riemann sum and clearly show the steps of your simplification.

$f(x) = 3x^2 - 1$

$[a,b] = [1,2]$

$\Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$

$x_i = a + i\Delta x = 1 + i\left(\frac{1}{n}\right)$

$f(x_i) = 3x_i^2 - 1 = 3\left(1 + \frac{i}{n}\right)^2 - 1 = 3\left(1 + \frac{2i}{n} + \frac{i^2}{n^2}\right) - 1$

$= 2 + \frac{6}{n} i + \frac{3}{n^2} i^2$

$\therefore \int_1^2 (3x^2 - 1)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(2 + \frac{6}{n} i + \frac{3}{n^2} i^2\right)\left(\frac{1}{n}\right)$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2}{n} + \frac{6}{n^2} i + \frac{3}{n^3} i^2\right)$

$= \lim_{n\to\infty} \left[\left(\sum_{i=1}^{n} \frac{2}{n}\right) + \frac{6}{n^2}\left(\sum_{i=1}^{n} i\right) + \frac{3}{n^3}\left(\sum_{i=1}^{n} i^2\right)\right]$

$= \lim_{n\to\infty} \left[\frac{2}{\cancel{n}} \cdot \cancel{n} + \frac{6}{n^2} \cdot \frac{n(n+1)}{2} + \frac{3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}\right]$

$= \lim_{n\to\infty} \left[2 + \frac{\cancel{6}^{3}}{\cancel{2}} \cdot \frac{\cancel{n}}{\cancel{n}} \cdot \frac{n+1}{n} + \frac{\cancel{3}^{1}}{\cancel{6}\,2} \cdot \frac{\cancel{n}}{\cancel{n}} \cdot \underbrace{\frac{n+1}{n}}_{\to 1} \cdot \underbrace{\frac{2n+1}{n}}_{\to 2}\right]$

$= 2 + 3 + 1$

$= \boxed{6}$

[10]

Question 4: The daytime temperature T over a 12 hour period is modelled by the function

$$T(t) = 10 + 6\sin\left(\frac{\pi t}{12}\right)$$

where T is in degrees celsius and t is in hours, and where $t = 0$ corresponds to 6:00 a.m. Determine the average temperature over the 12 hour period from 6:00 a.m. to 6:00 p.m.

$$T_{ave} = \frac{1}{12-0}\int_0^{12} 10 + 6\sin\left(\frac{\pi t}{12}\right) dt$$

$$= \frac{1}{12}\left[10t - \frac{6\cos\left(\frac{\pi t}{12}\right)}{(\pi/12)}\right]_0^{12}$$

$$= \frac{1}{12}\left[10t - \frac{72}{\pi}\cos\left(\frac{\pi t}{12}\right)\right]_0^{12}$$

$$= \frac{1}{12}\left[\left((10)(12) - \frac{72}{\pi}\cos\left(\frac{\pi\cdot 12}{12}\right)\right) - \left(0 - \frac{72}{\pi}\cos(0)\right)\right]$$

$$= \frac{1}{12}\left[120 + \frac{72}{\pi} + \frac{72}{\pi}\right] = \boxed{10 + \frac{12}{\pi} \text{ degrees}}$$

[5]

Question 5: Determine the following:

(i) $\displaystyle\int \frac{3x^2 - 2x + 1}{x}\, dx = \int 3x - 2 + \frac{1}{x}\, dx$

$$= \boxed{\frac{3x^2}{2} - 2x + \ln|x| + C}$$

[2]

(ii) $\displaystyle\int_0^{\pi/4} \sec^2(x) - \frac{4}{\pi}\, dx = \left[\tan(x) - \frac{4}{\pi}x\right]_0^{\pi/4}$

$$= \left(\tan\left(\frac{\pi}{4}\right) - \frac{4}{\pi}\cdot\frac{\pi}{4}\right) - \left(\tan(0) - \frac{4}{\pi}\cdot 0\right)$$

$$= (1-1) - (0-0)$$

$$= \boxed{0}$$

[3]

Question 6: Determine

$$I = \frac{1}{2}\int_{-1}^{1} 2xe^{x^2}\,dx$$

Let $u = x^2$, $du = 2x\,dx$

$x = -1 \Rightarrow u = (-1)^2 = 1$

$x = 1 \Rightarrow u = 1^2 = 1$

$$\therefore I = \frac{1}{2}\int_1^1 e^u\,du = \boxed{0}$$

[5]

Question 7: Determine

$$I = \int \frac{\cos(\ln(x))}{x}\,dx$$

Let $u = \ln(x)$

$du = \frac{1}{x}dx$

$$\therefore I = \int \cos(u)\,du$$

$$= \sin(u) + C$$

$$= \boxed{\sin(\ln(x)) + C}$$

[5]

Question 8: Determine

$$I = \frac{1}{3}\int_0^1 3x^2(1+x^3)^5\,dx$$

Let $u = 1 + x^3$ $\quad x=0 \Rightarrow u=1$

$du = 3x^2\,dx$ $\quad x=1 \Rightarrow u=2$

$$\therefore I = \frac{1}{3}\int_1^2 u^5\,du$$

$$= \frac{1}{3}\left[\frac{u^6}{6}\right]_1^2$$

$$= \frac{1}{18}\left[2^6 - 1^6\right] = \frac{\cancel{63}^{\,7}}{\cancel{18}_{\,2}} = \boxed{\frac{7}{2}}$$

[5]

Question 9: Determine

$$I = \int x\sqrt{1+x}\,dx$$

Let $u = 1+x \Rightarrow x = u-1$

$du = dx$

$$\therefore I = \int (u-1)\,u^{1/2}\,du$$

$$= \int u^{3/2} - u^{1/2}\,du$$

$$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$$

$$= \boxed{\frac{2}{5}(1+x)^{5/2} - \frac{2}{3}(1+x)^{3/2} + C}$$

[5]