Question 1:

(a) Find the linear approximation $T_1(x)$ about $a = 1$ for $f(x) = x^{7/2}$.

$f(x) = x^{7/2}$; $f(1) = 1$

$f'(x) = \frac{7}{2}x^{5/2}$; $f'(1) = \frac{7}{2}$

$\therefore T_1(x) = f(a) + f'(a)(x-a)$

$= \boxed{1 + \frac{7}{2}(x-1)}$

[4]

(b) Use your result in part (a) to approximate $(0.9)^{\frac{7}{2}}$. (Express your answer as a single simplified fraction.)

$(0.9)^{7/2} = f(0.9) \approx T_1(0.9)$

$= 1 + \frac{7}{2}(0.9-1)$

$= 1 + \frac{7}{2}\left(\frac{-1}{10}\right)$

$= \frac{20-7}{20}$

$= \boxed{\frac{13}{20}}$

[2]

(c) Give an error bound for your approximation in part (b). (Again, express your answer as a single simplified fraction.)

$f'(z) = \frac{7}{2}z^{5/2} \Rightarrow f''(z) = \frac{35}{4}z^{3/2}$

$R_1(x) = \frac{f''(z)(x-a)^2}{2}$ where $a = 1$, $x = 0.9$, $0.9 < z < 1$

$\therefore |R_1(0.9)| = \left|\frac{1}{2} \cdot \frac{35}{4}z^{3/2} \cdot (0.9-1)^2\right|$

$\leq \left|\frac{1}{2} \cdot \frac{35}{4} \cdot 1^{3/2} \cdot \frac{1}{100}\right|$

$= \frac{35}{800} = \boxed{\frac{7}{160}}$

[4]

Question 2:

(a) Find the Taylor polynomial of degree 3 for $f(x) = \cos(2+2x)$ at $a = -1$.

$f(x) = \cos(2+2x)$; $f(-1) = \cos(0) = 1$

$f'(x) = -2\sin(2+2x)$; $f'(-1) = 0$

$f''(x) = -4\cos(2+2x)$; $f''(-1) = -4$

$f'''(x) = 8\sin(2+2x)$; $f'''(-1) = 0.$

$$\therefore T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$= 1 + 0\cdot(x+1) - \frac{4}{2}(x+1)^2 + \frac{0}{3!}(x+1)^3$$

$$= \boxed{1 - 2(x+1)^2}$$

[5]

(b) Suppose $T_3(x)$ in part (a) is used to approximate $f(-3/2)$. Give an error bound on the approximation. Express your answer as a single simplified fraction. (Note: you are not being asked to find the approximation to $f(-3/2)$ here, but only the error bound associated with the approximation.)

$f^{(4)}(z) = 16\cos(2+2z)$, $-\frac{3}{2} < z < -1$.

$\therefore |f^4(z)| \leq 16\cdot 1 = 16.$

$$|R_4(-\tfrac{3}{2})| = \left| \frac{f^4(z)}{4!}(x-a)^4 \right|$$

$$\leq \left| \frac{16}{4!}\left(-\frac{3}{2}+1\right)^4 \right|$$

$$= \cancel{\frac{16}{4!}\cdot\left(\frac{-1}{2}\right)^4}$$

$$= \frac{1}{4!} = \boxed{\frac{1}{24}}$$

[5]

Question 3:

(a) Find the first four nonzero terms of the Maclaurin series for $f(x) = x^3 e^{-(x^2)}$.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$

$$\therefore e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \cdots$$

$$\therefore x^3 e^{-x^2} = \boxed{x^3 - x^5 + \frac{x^7}{2} - \frac{x^9}{3!} + \cdots}$$

[5]

(b) Determine $f^{(11)}(0)$ where again $f(x) = x^3 e^{-(x^2)}$. There is no need to simplify factorials (if any) in your final answer.

From (a): $x^3 e^{-x^2} = x^3 - x^5 + \frac{x^7}{2} - \frac{x^9}{3!} + \frac{x^{11}}{4!} - \frac{x^{13}}{5!} + \cdots$

$$\therefore \frac{f^{(11)}(0)\, x^{11}}{11!} = \frac{x^{11}}{4!}$$

$$\therefore \boxed{f^{11}(0) = \frac{11!}{4!}}$$

[3]

(c) Determine $f^{(12)}(0)$ where again $f(x) = x^3 e^{-(x^2)}$. There is no need to simplify factorials (if any) in your final answer.

From the series in (b),

$$\frac{f^{(12)}(0)\, x^{12}}{12!} = 0 \cdot x^{12}$$

so $\boxed{f^{(12)}(0) = 0}$

[2]

Question 4:

(a) Find the first four nonzero terms of the Maclaurin series for $f(x) = \dfrac{3x}{(1-x)^2}$ and state the open interval of convergence.
(Hint: think about the Maclaurin series for the derivative of $\dfrac{1}{1-x}$.)

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \quad |x| < 1$$

$$\therefore \frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\left[1 + x + x^2 + x^3 + \cdots\right]$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots, \quad |x| < 1$$

$$\therefore \frac{3x}{(1-x)^2} = \boxed{3x + 6x^2 + 9x^3 + 12x^4 + \cdots}, \quad \underbrace{|x| < 1}$$

$$\hookrightarrow \boxed{\therefore I = (-1, 1).}$$

[5]

(b) Find the the first four nonzero terms of the Taylor series for $h(x) = \dfrac{1}{1-x}$ about $a = -2$ and state the open interval of convergence of the series. (Hint: one way is to use the series for $\dfrac{1}{1-x}$.)

$$\frac{1}{1-x} = \frac{1}{1-(x+2)+2}$$

$$= \frac{1}{3-(x+2)}$$

$$= \frac{1}{3}\left[\frac{1}{1-\left[\frac{1}{3}(x+2)\right]}\right]$$

$$= \frac{1}{3}\left[1 + \frac{1}{3}(x+2) + \left(\frac{1}{3}(x+2)\right)^2 + \left(\frac{1}{3}(x+2)\right)^3 + \cdots\right]$$

$$= \boxed{\frac{1}{3} + \frac{1}{3^2}(x+2) + \frac{1}{3^3}(x+2)^2 + \frac{1}{3^4}(x+2)^3 + \cdots}$$

For I: $\left|\frac{1}{3}(x+2)\right| < 1 \Rightarrow |x+2| < 3 \Rightarrow \boxed{I = (-5, 1)}$

[5]

Question 5: Find the first three nonzero terms of the Maclaurin series for $f(x) = \cos(x) \cdot \ln(1-x)$.

$\cos(x) \cdot \ln(1-x)$

$= \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right]\left[-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\right]$

degree 0: no terms.

degree 1: $(1)(-x) = -x$

degree 2: $(1)\left(-\frac{x^2}{2}\right) = \frac{-x^2}{2}$

degree 3: $(1)\left(\frac{-x^3}{3}\right) + \left(\frac{-x^2}{2!}\right)(-x) = -\frac{x^3}{3} + \frac{x^3}{2} = \frac{x^3}{6}$

$\boxed{\therefore\ f(x) = -x - \frac{x^2}{2} + \frac{x^3}{6} + \cdots}$

[5]

Question 6: Evaluate the following limit and state your answer as a single simplified fraction:

$$\lim_{x\to0} \frac{e^{-(x^2)} + x^2 - 1}{\sin(3x^4)}$$

$= \lim_{x\to0} \frac{\left[\cancel{1} - \cancel{x^2} + \frac{x^4}{2} - \frac{x^6}{3!} + \cdots\right] + \cancel{x^2} - \cancel{1}}{3x^4 - \frac{(3x^4)^3}{3!} + \cdots}$

$= \lim_{x\to0} \frac{\cancel{x^4}\left[\frac{1}{2} - \frac{x^2}{3!} + \cdots\right]}{\cancel{x^4}\left[3 - \frac{27}{3!}x^8 + \cdots\right]}$

$= \frac{\left(\frac{1}{2}\right)}{3}$

$= \boxed{\frac{1}{6}}$

[5]