Question 1: Determine $\int \frac{2x^2+6x+7}{x(\underbrace{x^2+4x+7}_{\text{irreducible}})} = I$

$$\frac{2x^2+6x+7}{x(x^2+4x+7)} = \frac{A}{x} + \frac{Bx+C}{x^2+4x+7}$$

$$= \frac{A(x^2+4x+7)+(Bx+C)x}{x(x^2+4x+7)}$$

$$= \frac{(A+B)x^2+(4A+C)x+7A}{x(x^2+4x+7)}$$

$\therefore$ ① $A+B = 2$

② $4A+C = 6$

③ $7A = 7 \Rightarrow \boxed{A=1}$, $\boxed{C=2}$ from ②, $\boxed{B=1}$ from ①.

$$\therefore I = \int \frac{1}{x} + \frac{x+2}{x^2+4x+7}\,dx$$

$$= \int \frac{1}{x}\,dx + \frac{1}{2}\int \underbrace{\frac{2x+4}{x^2+4x+7}\,dx}$$

$u = x^2+4x+7$

$du = (2x+4)dx$

$$= \boxed{\ln|x| + \frac{1}{2}\ln|x^2+4x+7| + C}$$

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Question 2:

(a) Use T_4, the trapezoid rule on four subintervals, to approximate the integral $\int_0^2 \sin^2(\pi x)\,dx$.

$\Delta x = \frac{2-0}{4} = \frac{1}{2}$;

$0 = x_0$, $\frac{1}{2} = x_1$, $1 = x_2$, $\frac{3}{4} = x_3$, $2 = x_4$

$f(x) = \sin^2(\pi x)$.

$$T_4 = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4) \right]$$

$$= \frac{(1/2)}{2}\left[\underbrace{\sin^2(0)}_{0} + 2\underbrace{\sin^2\left(\frac{\pi}{2}\right)}_{1} + 2\underbrace{\sin^2(\pi)}_{0} + 2\underbrace{\sin^2\left(\frac{3\pi}{2}\right)}_{1} + \underbrace{\sin^2(2\pi)}_{0} \right]$$

$$= \left(\frac{1}{4}\right)\left[2+2\right]$$

$$= \boxed{1}$$

[5]

(b) Determine an error bound in your approximation T_4 in part (a). Recall, the error in using the trapezoid rule to approximate $\int_a^b f(x)\,dx$ is at most $\frac{K(b-a)^3}{12n^2}$, where $|f''(x)| \le K$ on $[a, b]$.

Hint: to determine K, you may find it easier if you think $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$.

$$f(x) = \sin^2(\pi x) = \frac{1}{2} - \frac{\cos(2\pi x)}{2}$$

$$f'(x) = \frac{\sin(2\pi x)}{\cancel{2}} \cdot \cancel{2}\pi = \pi\sin(2\pi x)$$

$$f''(x) = \pi\cos(2\pi x)\,(2\pi) = 2\pi^2\cos(2\pi x)$$

$$\therefore\ |f''(x)| = |2\pi^2\cos(2\pi x)| \le 2\pi^2 = K.$$

Using $K = 2\pi^2$, $a=0$, $b=2$, $n=4$:

$$|E_{T_4}| \le \frac{2\pi^2(2-0)^3}{12\cdot 4^2} = \boxed{\frac{\pi^2}{12}}$$

[5]

Question 3:

(a) Evaluate the following improper integral, making proper use of any required limits:

$$\int_0^{\ln 2} \frac{e^x}{\sqrt{e^x-1}}\,dx$$

$$= \lim_{b\to 0^+} \int_b^{\ln 2} e^x (e^x-1)^{-\frac{1}{2}}\,dx \quad \left\{ \begin{array}{l} u = e^x-1 \\ du = e^x dx \end{array} \right.$$

$$= \lim_{b\to 0^+} \left[2(e^x-1)^{\frac{1}{2}} \right]_b^{\ln 2}$$

$$= \lim_{b\to 0^+} \left[\underbrace{2(\cancel{e^{\ln 2}}_{2}-1)^{1/2}}_{=2} - \underbrace{2(e^b-1)^{1/2}}_{\to 0} \right]$$

$$= \boxed{2}$$

[5]

(b) Use the comparison theorem to decide if the following integral converges or diverges:

$$\int_1^{\infty} \frac{e^{-x}}{x^2+\sqrt{x}+1}\,dx$$

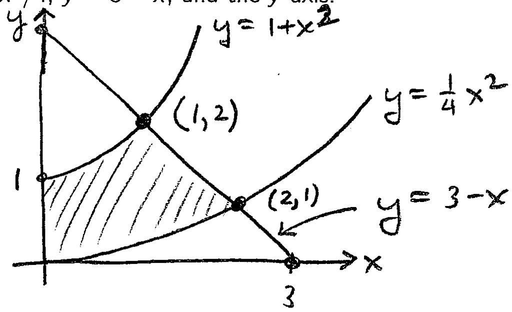
on $[1,\infty)$, $\dfrac{e^{-x}}{x^2+\sqrt{x}+1} \leq \dfrac{1}{x^2}$

Since $\int_1^\infty \frac{1}{x^2}\,dx$ converges (p-integral, $p=2>1$),

so does $\int_1^\infty \dfrac{e^{-x}}{x^2+\sqrt{x}+1}\,dx$ by the comparison theorem.

[5]

Question 4: Determine the area of the region in the first quadrant that is enclosed by the curves $y = 1 + x^2$, $y = x^2/4$, $y = 3 - x$, and the y-axis.



$$A = \int_0^1 (1+x^2) - \left(\tfrac{1}{4}x^2\right) dx + \int_1^2 (3-x) - \left(\tfrac{1}{4}x^2\right) dx$$

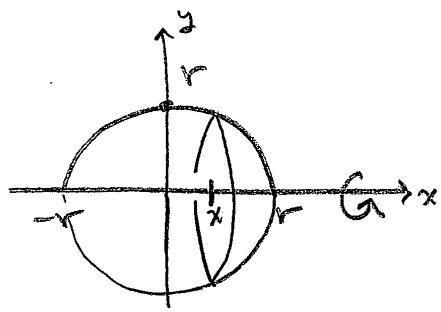
$$= \left[x + \tfrac{3}{4}\tfrac{x^3}{3}\right]_0^1 + \left[3x - \tfrac{x^2}{2} - \tfrac{1}{12}x^3\right]_1^2$$

$$= 1 + \tfrac{1}{4} + \left(6 - 2 - \tfrac{2}{3}\right) - \left(3 - \tfrac{1}{2} - \tfrac{1}{12}\right)$$

$$= \frac{12 + 3 + 48 - 8 - 36 + 6 + 1}{12} = \boxed{\frac{13}{6}}$$

[5]

Question 5: The curve $y = \sqrt{r^2 - x^2}$ over $-r \le x \le r$ represents the top half of a circle of radius r centred at $(0, 0)$. Rotating the curve about the x-axis forms a sphere of radius r. Use the disk method to determine the volume of the sphere.



$$V = \int_{-r}^{r} \pi \left(\sqrt{r^2 - x^2}\right)^2 dx$$

$$= \int_{-r}^{r} \pi (r^2 - x^2)\, dx$$

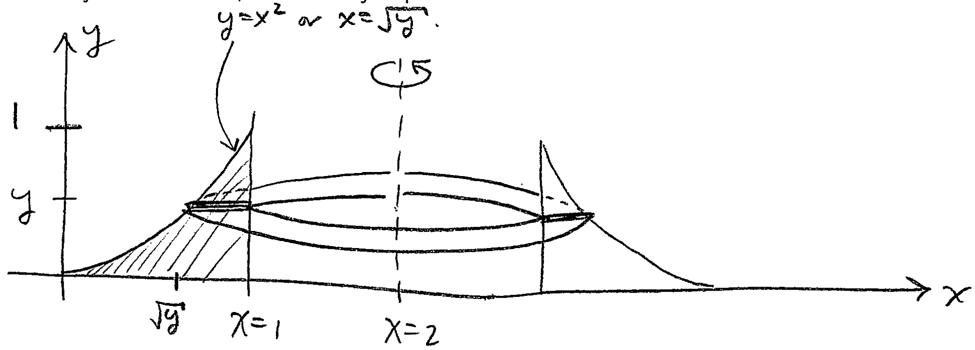
$$= \pi \left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r}$$

$$= \pi \left[\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 + \frac{r^3}{3}\right)\right]$$

$$= \boxed{\frac{4}{3}\pi r^3}$$

[5]

Question 6: The region in the first quadrant that is bounded by $y = x^2$, the vertical line $x = 1$ and the x-axis is rotated about the vertical line $x = 2$. Determine the volume of the resulting solid. Use either the washer method or cylindrical shells, whichever you prefer.



Washer:

$$V = \int_{y=0}^{y=1} \pi\left[(2-\sqrt{y})^2 - (2-1)^2\right] dy$$

$$= \int_0^1 \pi\,(3 - 4y^{1/2} + y)\, dy$$

$$= \pi\left[3y - \frac{8}{3}y^{3/2} + \frac{y^2}{2}\right]_0^1$$

$$= \pi\left(3 - \frac{8}{3} + \frac{1}{2}\right)$$

$$= \frac{\pi}{6}(18 - 16 + 3)$$

$$= \boxed{\frac{5\pi}{6}}$$

Cylindrical Shells:

$$V = \int_{x=0}^{x=1} 2\pi(2-x)\,x^2\, dx$$

$$= 2\pi\int_0^1 2x^2 - x^3\, dx$$

$$= 2\pi\left[\frac{2}{3}x^3 - \frac{x^4}{4}\right]_0^1$$

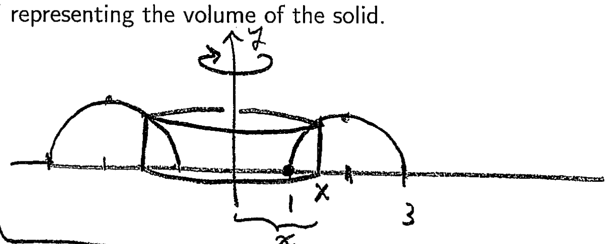
$$= 2\pi\left(\frac{2}{3} - \frac{1}{4}\right)$$

$$= \cancel{2}\pi\left(\frac{5}{\cancel{12}_6}\right)$$

$$= \boxed{\frac{5\pi}{6}}$$

[5]

Question 7: The curve $y = \sqrt{1-(x-2)^2}$ over $1 \le x \le 3$ represents the top half of a circle of radius 1 centred at $(2, 0)$. If the region between the curve and x-axis is rotated about the y-axis, the resulting solid is the top half of a torus (a torus is essentially a mathematical doughnut, or bagel). Set up BUT DO NOT EVALUATE the integral representing the volume of the solid.



$$\boxed{V = \int_1^3 2\pi x\sqrt{1-(x-2)^2}\, dx}$$

[5]