Question 1: Use the definition of the definite integral in the form

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

to evaluate

$$\int_1^2 (2x^2+1)\,dx$$

Carefully set up the Riemann sum and clearly show the steps of your simplification.

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

$$x_i = a + i\Delta x = 1 + i\left(\frac{1}{n}\right)$$

$$f(x_i) = 2x_i^2 + 1 = 2\left(1+\frac{i}{n}\right)^2 + 1$$

$$= 2\left(1 + \frac{2i}{n} + \frac{i^2}{n^2}\right) + 1$$

$$= 3 + \frac{4}{n} i + \frac{2}{n^2} i^2$$

$$\int_1^2 (2x^2+1)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(3 + \frac{4}{n} i + \frac{2}{n^2} i^2\right)\left(\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} \left[\sum_{i=1}^{n}\left(\frac{3}{n}\right) + \sum_{i=1}^{n}\left(\frac{4}{n^2} i\right) + \sum_{i=1}^{n}\left(\frac{2}{n^3} i^2\right) \right]$$

$$= \lim_{n\to\infty} \left[\left(\frac{3}{n}\right)\left(\sum_{i=1}^{n} 1\right) + \left(\frac{4}{n^2}\right)\left(\sum_{i=1}^{n} i\right) + \left(\frac{2}{n^3}\right)\left(\sum_{i=1}^{n} i^2\right) \right]$$

$$= \lim_{n\to\infty} \left[\left(\frac{3}{\cancel{n}}\right)(\cancel{n}) + \left(\frac{\cancel{4}^{\,2}}{\cancel{2}}\right)\left(\frac{\cancel{n}}{\cancel{n}}\right)\left(\frac{n+1}{n}\right) + \left(\frac{\cancel{2}^{\,1}}{\cancel{6}_{\,3}}\right)\left(\frac{\cancel{n}}{\cancel{n}}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) \right]$$

$$= 3 + 2 + \left(\frac{1}{3}\right)(2)$$

$$= \boxed{\frac{17}{3}}$$

Check: $\int_1^2 (2x^2+1)\,dx = \left[\frac{2}{3}x^3 + x\right]_1^2$

$$= \left(\frac{2}{3}\right)(8) + 2 - \frac{2}{3} - 1$$

$$= \frac{16+6-2-3}{3}$$

$$= \boxed{\frac{17}{3}}$$

[10]

Question 2: A town's population is growing at a rate given by $P'(t) = 1200e^{t/100}$ people per year, where $t = 0$ corresponds to the present. Determine the change in population over the next 100 years.

$$P(100) - P(0) = \int_0^{100} 1200\, e^{t/100}\, dt$$

$$= (1200)(100)\left[e^{t/100}\right]_0^{100}$$

$$= \boxed{120000\,(e-1) \text{ people.}}$$

[5]

Question 3: Evaluate the following definite integrals:

(a) $\int_0^1 (1-x)^{99}\, dx \quad = \left[-\frac{(1-x)^{100}}{100}\right]_0^1$

$$= 0 + \frac{1}{100}$$

$$= \boxed{\frac{1}{100}}$$

[2]

(b) $\int_1^4 \frac{\sqrt{x} - 3x^2 + 1}{x}\, dx = \int_1^4 x^{-\frac{1}{2}} - 3x + \frac{1}{x}\, dx$

$$= \left[2x^{\frac{1}{2}} - \frac{3}{2}x^2 + \ln|x|\right]_1^4$$

$$= \left[2(4)^{\frac{1}{2}} - \frac{3}{2}(4)^2 + \ln|4|\right] - \left[2(1)^{\frac{1}{2}} - \frac{3}{2}(1)^2 + \underbrace{\cancel{\ln|1|}}_{0}\right]$$

$$= 4 - 24 + \ln|4| - 2 + \frac{3}{2} + 0$$

$$= \boxed{\ln|4| - \frac{41}{2}}$$

[3]

Question 4: Find the following integrals (Integration by Substitution):

(a) $\frac{1}{3}\int \cancel{3}x^2\sqrt{1+x^3}\,dx$

$u = 1+x^3$

$du = 3x^2 dx$

$= \frac{1}{3}\int u^{\frac{1}{2}}\,du$

$= \frac{1}{3}\,\frac{2}{3}\,u^{3/2} + C$

$= \boxed{\frac{2}{9}(1+x^3)^{3/2} + C}$

[3]

(b) $\int \frac{\cos(\ln(2x))}{x}\,dx$

$u = \ln(2x)$

$du = \frac{1}{\cancel{2}x}\cancel{2}\,dx$

$= \int \cos(u)\,du$

$= \text{sen}(u) + C$

$= \boxed{\text{sen}(\ln(2x)) + C}$

[3]

(c) $\int \frac{x}{1+x^4}\,dx = \frac{1}{2}\int \frac{2x}{1+(x^2)^2}\,dx$

$u = x^2$

$du = 2x\,dx$

$= \frac{1}{2}\int \frac{1}{1+u^2}\,du$

$= \frac{1}{2}\arctan(u) + C$

$= \boxed{\frac{1}{2}\arctan(x^2) + C}$

[4]

Question 5: Find the following integrals (Integration by Parts):

(a) $\int x^2 e^{2x}\,dx$

$u = x^2 \quad dv = e^{2x}dx$

$du = 2x \quad v = \frac{e^{2x}}{2}$

$= uv - \int v\,du$

$= \frac{x^2e^{2x}}{2} - \int x e^{2x}\,dx$ $\quad u = x \quad dv = e^{2x}dx$, $\quad du = dx \quad v = \frac{e^{2x}}{2}$

$= \frac{x^2e^{2x}}{2} - \left[uv - \int v\,du\right]$

$= \frac{x^2e^{2x}}{2} - \left[\frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2}\,dx\right]$

$= \boxed{\frac{x^2e^{2x}}{2} - \frac{xe^{2x}}{2} + \frac{e^{2x}}{4} + C}$

[5]

(b) $\int \arctan(1/x)\,dx$

$u = \arctan\left(\frac{1}{x}\right) \quad dv = dx$

$du = \frac{1}{1+\left(\frac{1}{x}\right)^2}\cdot\left(\frac{-1}{x^2}\right)dx \quad v = x$

$= \frac{-1}{1+x^2}\,dx$

$= uv - \int v\,du$

$= x\arctan\left(\frac{1}{x}\right) - \int \frac{-x}{1+x^2}\,dx$

$= x\arctan\left(\frac{1}{x}\right) + \frac{1}{2}\underbrace{\int \frac{2x}{1+x^2}\,dx}_{w = 1+x^2,\ dw = 2x\,dx}$

$= \boxed{x\arctan\left(\frac{1}{x}\right) + \frac{1}{2}\ln\left|1+x^2\right| + C}$

[5]

Question 6: Determine $\int \tan^5(x)\sec^4(x)\,dx$.

$$= \int \tan^4(x)\sec^3(x) \quad \sec(x)\tan(x)\,dx$$

$$= \int (\sec^2(x)-1)^2 \sec^3(x)\ \sec(x)\tan(x)\,dx \quad \left\{ \begin{array}{l} u = \sec(x) \\ du = \sec(x)\tan(x)\,dx \end{array} \right.$$

$$= \int (u^2-1)^2 u^3\,du$$

$$= \int u^7 - 2u^5 + u^3\,du$$

$$= \frac{u^8}{8} - 2\frac{u^6}{6} + \frac{u^4}{4} + C$$

$$= \boxed{\frac{\sec^8(x)}{8} - \frac{\sec^6(x)}{3} + \frac{\sec^4(x)}{4} + C}$$

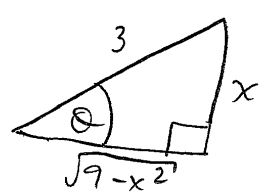
[5]

Question 7: Determine $\underbrace{\int \frac{9}{x^2\sqrt{9-x^2}}\,dx}_{I}$. $\left\{ \begin{array}{l} x = 3\sin\theta \\ dx = 3\cos\theta\,d\theta \end{array} \right.$ $\longleftarrow \sin\theta = \frac{x}{3}$

$$\therefore I = \int \frac{\cancel{9}\cdot 3\cos\theta}{\cancel{9}\sin^2\theta\sqrt{9-9\sin^2\theta}}\,d\theta$$

$$= 3\int \frac{\cos\theta}{\sin^2\theta\sqrt{9\cos^2\theta}}\,d\theta$$

$$= \cancel{3}\int \frac{\cancel{\cos\theta}}{\sin^2\theta\cdot\cancel{3}\cdot\cancel{\cos\theta}}\,d\theta$$



$$= \int \csc^2\theta\,d\theta$$

$$= -\cot\theta + C$$

$$= \boxed{-\frac{\sqrt{9-x^2}}{x} + C}$$

[5]