Question 1:

(a)[5] Determine an equation of the tangent line to the curve

$$xy - y^2 = 1 - \frac{2}{y}$$

at the point $(0, 1)$.

$$\frac{d}{dx}\left[xy - y^2\right] = \frac{d}{dx}\left[1 - \frac{2}{y}\right]$$

$$(1)y + xy' - 2yy' = \frac{2}{y^2}y'$$

at $(0,1)$: $\quad 1 + 0 - 2y' = 2y'$

$$4y' = 1$$

$$y' = \frac{1}{4}$$

$$\therefore \quad y - 1 = \frac{1}{4}(x - 0)$$

or $\boxed{y = \frac{1}{4}x + 1}$

(b)[5] Use logarithmic differentiation to determine y' where $y = (\sqrt{x} - 1)^x$.

$$\ln y = x \ln(\sqrt{x} - 1)$$

$$\frac{1}{y}y' = \ln(\sqrt{x} - 1) + \frac{x}{\sqrt{x} - 1}\left(\frac{1}{2\sqrt{x}}\right)$$

$$\boxed{y' = (\sqrt{x} - 1)^x\left[\ln(\sqrt{x} - 1) + \frac{\sqrt{x}}{2(\sqrt{x} - 1)}\right]}$$

Question 2:

(a)[3] Differentiate $y = e^{\cos x} - \ln(\sin x)$

$$\boxed{y' = -e^{\cos x} \cdot \sin x - \frac{\cos x}{\sin x}}$$

(b)[3] Differentiate $y = 7^{x \sec x}$

$$\boxed{y' = 7^{x \sec x} \cdot \ln(7) \cdot [\sec x + x \sec x \tan x]}$$

(c)[4] Determine $f''(0)$ where $f(x) = e^x \ln(1+x)$. (Simplify your final answer.)

$$f'(x) = e^x \ln(1+x) + \frac{e^x}{1+x}$$

$$f''(x) = e^x \ln(1+x) + \frac{e^x}{1+x} + \frac{(1+x)e^x - e^x}{(1+x)^2}$$

$$f''(0) = \cancel{1 \cdot 0}^{\,0} + \frac{1}{1} + \cancel{\frac{1 \cdot 1 - 1}{(1+0)^2}}^{\,0}$$

$$= \boxed{1}$$

Question 3:

(a)[4] Determine $\lim\limits_{x \to 4^-} \dfrac{\ln(4-x)}{x}$.

(If the limit does not exist but is ∞ or $-\infty$, state which with an explanation of your reasoning.)

As $x \to 4^-$, $4 - x \to 0^+$,

so $\ln(4-x) \to -\infty$,

so $\dfrac{\ln(4-x)}{x} \to -\infty$

$\therefore \lim\limits_{x \to 4^-} \dfrac{\ln(4-x)}{x} = \boxed{-\infty}$

(b)[6] Use a linear approximation to estimate $\dfrac{1}{1.01}$.

Here $f(x) = \dfrac{1}{x}$, $a = 1$

$f(a) = f(1) = 1.$

$f'(x) = \dfrac{-1}{x^2}$; $f'(a) = \dfrac{-1}{1^2} = -1.$

$\therefore L(x) = f(a) + f'(a)(x-a)$

$= 1 - 1(x-1).$

$\therefore \dfrac{1}{1.01} = f(1.01) \approx L(1.01)$

$= 1 - 1(1.01 - 1)$

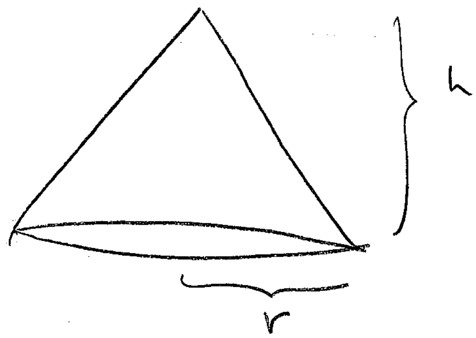
$= 1 - \dfrac{1}{100}$

$= \boxed{\dfrac{99}{100}}$

Question 4 (related rates) [10]:

Sand is falling onto a pile at a rate of 2 m^3 per minute. The pile of sand maintains the shape of a cone with height equal to the diameter of the base. How fast is the height of the sand pile increasing when the height is exactly 4 m? State units with your answer. (Recall: A cone of height h and base radius r has volume $V = \frac{\pi}{3} r^2 h$.)

$\frac{dV}{dt} = 2 \frac{m^3}{min}$



$h = 2r$ at all times t.

Find $\frac{dh}{dt}$ when $h = 4$.

$V = \frac{\pi}{3} r^2 h$

$V = \frac{\pi}{3} \left(\frac{h}{2}\right)^2 h = \frac{\pi}{12} h^3.$

$\frac{dV}{dt} = \frac{\pi}{12} 3h^2 \frac{dh}{dt}$

When $h = 4$:

$2 = \left(\frac{\pi}{12}\right)(3)(4^2) \frac{dh}{dt}$

$\therefore \frac{dh}{dt} = \frac{24}{48\pi} = \frac{1}{2\pi} \frac{m}{min}$

$\therefore$ heigeht is increasing at $\frac{1}{2\pi} \frac{m}{min}$.

Question 5: For this question use the function $f(x) = (x-2)^2 e^x$

(a)[7] Determine the intervals of increase and decrease of $f(x)$. State a clear conclusion.

Domain of f is $(-\infty, \infty)$.

$$f'(x) = 2(x-2)e^x + (x-2)^2 e^x$$
$$= e^x (x-2)(2 + x - 2)$$
$$= e^x x(x-2).$$

- $f'(x)=0$? $x=0$, $x=2$.
- $f'(x)$ not exist? no such x in domain.

		0		2	
test values:	-1		1		3
$f'(x) = e^x x(x-2)$:	$+$	0	$-$	0	$+$
$f(x) = (x-2)^2 e^x$:	$\nearrow$	4	$\searrow$	0	$\nearrow$

(e^x positive)

$\therefore$ f is increasing on $(-\infty, 0) \cup (2, \infty)$

f is decreasing on $(0, 2)$.

(b)[3] State the relative extrema of $f(x)$.

f has a rel. max. of 4 at $x=0$.

f has a rel. min. of 0 at $x=2$.