Question 1: Evaluate the following limits, if they exist. If a limit does not exist but is ∞ or $-\infty$, state which with an explanation of your answer.

(a)[3] $\displaystyle\lim_{x\to-\infty} \frac{1-2x^2-x^4}{5+x-3x^4}$ $= \displaystyle\lim_{x\to-\infty} \frac{x^4\left(\frac{1}{x^4}-\frac{2}{x^2}-1\right)}{x^4\left(\frac{5}{x^4}+\frac{1}{x^3}-3\right)}$

$$= \frac{-1}{-3}$$

$$= \boxed{\frac{1}{3}}$$

(b)[4] $\displaystyle\lim_{x\to\infty} \sqrt{x+1}-\sqrt{x-1}$ $= \displaystyle\lim_{x\to\infty} \frac{\sqrt{x+1}-\sqrt{x-1}}{1}\cdot\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}$

$$= \lim_{x\to\infty} \frac{x+1-(x-1)}{\sqrt{x+1}+\sqrt{x-1}}$$

$$= \lim_{x\to\infty} \frac{2}{\sqrt{x+1}+\sqrt{x-1}} = \boxed{0}$$

(c)[3] $\displaystyle\lim_{x\to\pi} \frac{\cos x}{\sin^2 x}$

As $x\to\pi$, $\cos(x)\to-1$ while $\sin^2(x)\to 0^+$

$\therefore \displaystyle\lim_{x\to\pi} \frac{\cos x}{\sin^2 x} = \boxed{-\infty}$

Question 2:

(a)[4] Determine all values of x at which tangent lines to $y = 2x^3 + 3x^2 - 9x + 1$ are parallel to the line $y = 3x - 7$.

$y = 3x - 7$ has slope 3, so solve $\frac{d}{dx}[2x^3 + 3x^2 - 9x + 1] = 3$

$\therefore 6x^2 + 6x - 9 = 3$

$6[x^2 + x - 2] = 0$

$6(x-1)(x+2) = 0$

$$\boxed{\therefore \ x = 1, \ x = -2}$$

(b)[6] A particle has displacement at time t seconds of $s(t) = t^3 - 12t^2 + 36t$ metres, where $t = 0$ corresponds to the present.

(i) At what times t is the particle at rest?

Solve $\quad v(t) = s'(t) = 0$:

$3t^2 - 24t + 36 = 0$

$3[t^2 - 8t + 12] = 0$

$3(t-2)(t-6) = 0$

$$\boxed{\therefore \ t = 2, \ t = 6}$$

(ii) What is the velocity when the acceleration is -6 m/s²?

$a(t) = s''(t) = 6t - 24$.

$a(t) = -6 \implies 6t - 24 = -6$

$\implies t = \frac{18}{6} = 3$ s.

$\therefore v(3) = 3(3^2) - 24(3) + 36 = \boxed{-9 \frac{m}{s}}$

Question 3:

(a)[3] Compute $f'(\pi)$ if $f(x) = \dfrac{\cos x}{1 - \sin x}$.

$$f'(x) = \frac{(1-\sin x)(-\sin x) - \cos x\,(-\cos x)}{(1-\sin x)^2}$$

$$f'(\pi) = \frac{(1-\sin\pi)(-\sin\pi) - \cos\pi\,(-\cos\pi)}{(1-\sin\pi)^2}$$

$$= \boxed{1}$$

(b)[3] Differentiate $g(x) = \dfrac{\pi^2 - x^2}{x} = \pi^2 x^{-1} - x$

$$g'(x) = \boxed{-\pi^2 x^{-2} - 1}$$

$$\text{or} = \frac{-\pi^2}{x^2} - 1$$

(c)[4] Determine an equation of the tangent line to $y = \sqrt{x} + \sqrt[5]{x}$ at the point where $x = 1$.

$$x=1 \Rightarrow y = \sqrt{1} + \sqrt[5]{1} = 2 \text{ , so } (1,2) \text{ is a point on the line.}$$

Slope of line is $\dfrac{d}{dx}\left[x^{\frac{1}{2}} + x^{\frac{1}{5}}\right]\Big|_{x=1} = \left[\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{5}x^{-\frac{4}{5}}\right]\Big|_{x=1}$

$$= \frac{1}{2} + \frac{1}{5}$$

$$= \frac{7}{10}$$

$\therefore$ Equation of tangent line is $\boxed{y - 2 = \dfrac{7}{10}(x-1)}$

Question 4:

(a)[4] Determine $f''(0)$ if $f(x) = \dfrac{4x}{\sqrt{x+1}} = 4x(x+1)^{-\frac{1}{2}}$

$$f'(x) = 4(x+1)^{-\frac{1}{2}} + 4x\left(\frac{-1}{2}\right)(x+1)^{-\frac{3}{2}}$$

$$= 4(x+1)^{-\frac{1}{2}} - 2x(x+1)^{-3/2}$$

$$f''(x) = (4)\left(\frac{-1}{2}\right)(x+1)^{-\frac{3}{2}} - 2(x+1)^{-3/2} - 2x\left(\frac{-3}{2}\right)(x+1)^{-5/2}$$

$$\therefore f''(0) = -2 - 2 - 0 = \boxed{-4}$$

(b)[3] Differentiate $g(x) = x\sec(1/x)$

$$g'(x) = (1)\sec\left(\frac{1}{x}\right) + x\sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right) \cdot \frac{d}{dx}\left[\frac{1}{x}\right]$$

$$= \boxed{\sec\left(\frac{1}{x}\right) + x\sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right)}$$

$$= \sec\left(\frac{1}{x}\right) - \sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)$$

(c)[3] Differentiate $y = (3t-1)^4(2t+1)^3$

$$\boxed{y' = 4(3t-1)^3(3)(2t+1)^3 + (3t-1)^4(3)(2t+1)^2(2)}$$

Question 5:

(a)[3]   Differentiate $y = \dfrac{1}{x^2 + \sqrt{3x+4}} = \left[x^2 + (3x+4)^{\frac{1}{2}} \right]^{-1}$

$$\therefore \quad y' = -\left[x^2 + (3x+4)^{\frac{1}{2}} \right]^{-2} \left[2x + \tfrac{1}{2}(3x+4)^{-\frac{1}{2}}(3) \right]$$

(b)[3]   Differentiate $y = \tan^2(\sin\theta)$

$$y' = 2\tan(\sin\theta) \cdot \sec^2(\sin\theta) \cdot \cos\theta$$

(c)[4]   Determine the values of the constants a and b so that the line $2x + y = b$ is tangent to the parabola $y = ax^2$ when $x = 2$.

$2x + y = b \implies y = -2x + b :$ Slope $m = -2$.

$\therefore \dfrac{d}{dx}\left[ax^2 \right]\Big|_{x=2} = -2 \implies 2ax\Big|_{x=2} = -2 \implies 4a = -2 \implies \boxed{a = \dfrac{-1}{2}}$

$\therefore$ parabola is $y = \left(\dfrac{-1}{2}\right)x^2$ and at $x=2$, $y = \left(\dfrac{-1}{2}\right)(2)^2 = -2$.

$\therefore$ Point $(2,-2)$ is on both $y = \left(\dfrac{-1}{2}\right)x^2$ and line $2x+y=b$.

$\therefore \quad 2(2) + (-2) = b \implies \boxed{b = 2}$