Question 1: Evaluate the following limits, if they exist. If a limit does not exist but is ∞ or $-\infty$, state which with an explanation of your answer.

(a)[3] $\lim_{x\to\pi} \frac{\cos x}{\sin^2 x}$

as $x \to \pi$, $\cos(x) \to -1$ while $\sin^2(x) \to 0^+$.

$\therefore \lim_{x\to\pi} \frac{\cos(x)}{\sin^2(x)} = \boxed{-\infty}$

(b)[3] $\lim_{x\to-\infty} \frac{1-2x^2-x^4}{5+x-3x^4}$

$$= \lim_{x\to-\infty} \frac{\cancel{x^4}\left(\frac{1}{x^4} - \frac{2}{x^2} - 1\right)}{\cancel{x^4}\left(\frac{5}{x^4} + \frac{1}{x^3} - 3\right)}$$

$$= \frac{-1}{-3}$$

$$= \boxed{\frac{1}{3}}$$

(c)[4] $\lim_{x\to\infty} \sqrt{x+1} - \sqrt{x-1}$

$$= \lim_{x\to\infty} \frac{\sqrt{x+1}-\sqrt{x-1}}{1} \cdot \frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}$$

$$= \lim_{x\to\infty} \frac{x+1-(x-1)}{\sqrt{x+1}+\sqrt{x-1}}$$

$$= \lim_{x\to\infty} \frac{2}{\sqrt{x+1}+\sqrt{x-1}} = \boxed{0}$$

Question 2:

(a)[6] A particle has displacement at time t seconds of $s(t) = t^3 - 12t^2 + 36t$ metres, where $t = 0$ corresponds to the present.

(i) At what times t is the particle at rest?

Solve $v(t) = s'(t) = 0$:

$$3t^2 - 24t + 36 = 0$$

$$3\left[t^2 - 8t + 12\right] = 0$$

$$3(t-2)(t-6) = 0$$

$$\boxed{t = 2,\ t = 6}$$

(ii) What is the displacement when the acceleration is -12 m/s^2?

$$a(t) = s''(t) = 6t - 24$$

$$a(t) = -12 \Rightarrow 6t - 24 = -12$$

$$\Rightarrow t = \frac{12}{6} = 2$$

$$\therefore s(2) = 2^3 - 12(2^2) + 36(2)$$

$$= \boxed{32 \text{ m}}$$

(b)[4] Determine all values of x at which tangent lines to $y = 2x^3 + 3x^2 - 9x + 1$ are parallel to the line $y = 3x - 7$.

$y = 3x - 7$ has slope 3, so solve $\frac{d}{dx}\left[2x^3 + 3x^2 - 9x + 1\right] = 3$:

$$6x^2 + 6x - 9 = 3$$

$$6\left[x^2 + x - 2\right] = 0$$

$$6(x-1)(x+2) = 0$$

$$\boxed{x = 1,\ x = -2}$$

Question 3:

(a)[4] Determine an equation of the tangent line to $y = \sqrt{x} + \sqrt[3]{x}$ at the point where $x = 1$.

At $x = 1$, $y = \sqrt{1} + \sqrt[3]{1} = 2$, so $(1,2)$ is a point on the line.

Slope of tangent line is $\frac{d}{dx}\left[x^{1/2} + x^{1/3}\right]\Big|_{x=1}$

$$= \left[\tfrac{1}{2}x^{-\frac{1}{2}} + \tfrac{1}{3}x^{-1/3}\right]\Big|_{x=1}$$

$$= \tfrac{1}{2} + \tfrac{1}{3} = \frac{5}{6}$$

$\therefore$ Equation of tangent line is $\boxed{y - 2 = \frac{5}{6}(x-1)}$

(b)[3] Differentiate $g(x) = \dfrac{\pi^2 - x^2}{x}$

$g(x) = \pi^2 x^{-1} - x$

$\therefore g'(x) = \boxed{-\pi^2 x^{-2} - 1}$

$\underline{\text{or}} = -\dfrac{\pi^2}{x^2} - 1$

(c)[3] Compute $f'(\pi)$ if $f(x) = \dfrac{\cos x}{1 - \sin x}$.

$$f'(x) = \frac{(1-\sin x)(-\sin x) - (\cos x)(0 - \cos x)}{(1-\sin x)^2}$$

$$f'(\pi) = \frac{(1-\sin\pi)\cancel{(-\sin\pi)}^{0} - (\cancel{\cos}^{-1}\pi)(-\cancel{\cos}^{-1}\pi)}{(1-\cancel{\sin\pi}_{0})^2}$$

$$= \boxed{1}$$

Question 4:

(a)[3] Differentiate $y = (3t-1)^4(2t+1)^3$

$$\boxed{y' = 4(3t-1)^3(3)(2t+1)^3 + (3t-1)^4(3)(2t+1)^2(2)}$$

(b)[3] Differentiate $g(x) = x\sec(1/x)$

$$\boxed{g'(x) = (1)\sec\left(\tfrac{1}{x}\right) + x\sec\left(\tfrac{1}{x}\right)\tan\left(\tfrac{1}{x}\right)\left(\tfrac{-1}{x^2}\right)}$$

(c)[4] Determine $f''(0)$ if $f(x) = \dfrac{4x}{\sqrt{x+1}} = 4x(x+1)^{-\frac{1}{2}}$

$$f'(x) = 4(x+1)^{-\frac{1}{2}} + 4x\left(-\tfrac{1}{2}\right)(x+1)^{-3/2}$$

$$= 4(x+1)^{-\frac{1}{2}} - 2x(x+1)^{-3/2}$$

$$f''(x) = 4\left(-\tfrac{1}{2}\right)(x+1)^{-3/2} - 2(x+1)^{-3/2} - 2x\left(-\tfrac{3}{2}\right)(x+1)^{-5/2}$$

$$\therefore f''(0) = -2 - 2 - 0$$

$$= \boxed{-4}$$

Question 5:

(a)[3] Differentiate $y = \tan^2(\sin\theta)$

$$\boxed{y' = 2\tan(\sin\theta)\cdot\sec^2(\sin\theta)\cdot\cos\theta}$$

(b)[3] Differentiate $y = \dfrac{1}{x^2+\sqrt{3x+4}}$ $= \left[x^2+(3x+4)^{\frac{1}{2}}\right]^{-1}$

$$\boxed{y' = -\left[x^2+(3x+4)^{\frac{1}{2}}\right]^{-2}\cdot\left[2x+\tfrac{1}{2}(3x+4)^{-\frac{1}{2}}(3)\right]}$$

(c)[4] Determine the values of the constants a and b so that the line $2x + y = b$ is tangent to the parabola $y = ax^2$ when $x = 2$.

$2x+y=b \Rightarrow y=-2x+b$: slope $m=-2$.

$\therefore \frac{d}{dx}\left[ax^2\right]_{x=2} = -2 \Rightarrow 2ax\big|_{x=2} = -2 \Rightarrow 4a=-2 \Rightarrow \boxed{a=-\frac{1}{2}}$

$\therefore$ parabola is $y=(-\frac{1}{2})x^2$ and at $x=2$ $y=(-\frac{1}{2})(2^2)=-2$.

$\therefore$ Point $(2,-2)$ is on both $y=(-\frac{1}{2})x^2$ and line $2x+y=b$.

$\therefore$ $2(2)+(-2)=b \Rightarrow \boxed{b=2}$