

2 Second Derivatives and Shapes of Curves

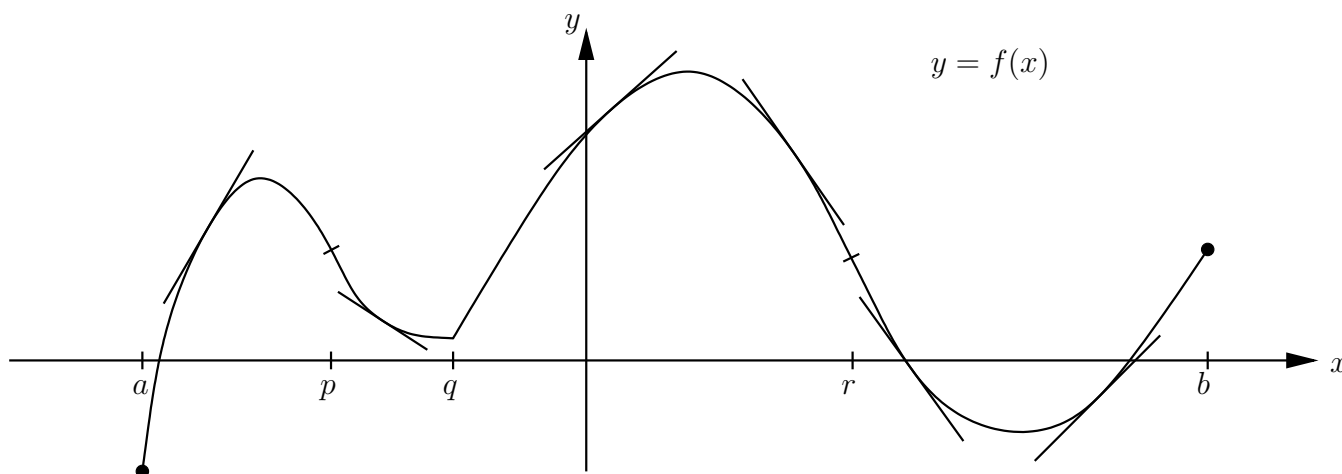
We have seen how $f'(x)$ gives us information about how a function increases and decreases, in particular:

- (i) If $f'(x) > 0$ on an interval, then f is increasing on that interval.
- (ii) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Now we examine what the second derivative $f''(x)$ tells us about f and its graph. First, recall

$$\begin{aligned} f''(x) &= \frac{d}{dx} [f'(x)] \\ &= \text{the rate of change of } f'(x) \\ &= \text{the slope of tangent lines to } y = f'(x) \end{aligned}$$

To see how this applies to the graph of $y = f(x)$, consider a general graph in which some tangent lines have been drawn:



Reading the graph left to right (as always!), notice:

- (i) Slopes of tangent lines decrease over the intervals (a, p) and (q, r) . Over these intervals tangent lines lie above the graph, and the graph itself bends downward.
- (ii) Slopes of tangent lines increase over the intervals (p, q) and (r, b) . Over these intervals tangent lines lie below the graph, and the graph itself bends upward.
- (iii) The transitions between increasing and decreasing tangent slopes, that is, transitions between bending trends, occur at $x = p$, $x = q$ and at $x = r$.

Now, extend what we learned about first derivatives to second derivatives:

$$\begin{aligned} f''(x) > 0 &\Rightarrow f'(x) \text{ is increasing} \Rightarrow \text{tangent slopes are increasing} \Rightarrow \text{graph of } y = f(x) \text{ bends upward} \\ f''(x) < 0 &\Rightarrow f'(x) \text{ is decreasing} \Rightarrow \text{tangent slopes are decreasing} \Rightarrow \text{graph of } y = f(x) \text{ bends downward} \end{aligned}$$

The manner in which a graph bends or curves is known as **concavity**, and this property is described by the second derivative.

2.1 Definitions

concave up: if the graph of f lies above all of its tangents on an interval, then the graph is said to be **concave up** on the interval. Think: the graph bends upwards.

concave down: if the graph of f lies below all of its tangents on an interval, then the graph is said to be **concave down** on the interval. Think: the graph bends downwards.

inflection point: an inflection point on the curve $y = f(x)$ is a point $(c, f(c))$ at which

- (i) f is continuous, and
- (ii) the graph of $y = f(x)$ changes concavity (i.e. changes from concave up to concave down or vice versa.)

So, referring to the graph above, we would say:

- f is concave down on (a, p) and (q, r) ;
- f is concave up on (p, q) and (r, b) ;
- f has inflection points at $(p, f(p))$, $(q, f(q))$ and $(r, f(r))$.

2.2 Concavity Test

Concavity as described by the second derivative is formalized in the **Concavity Test**:

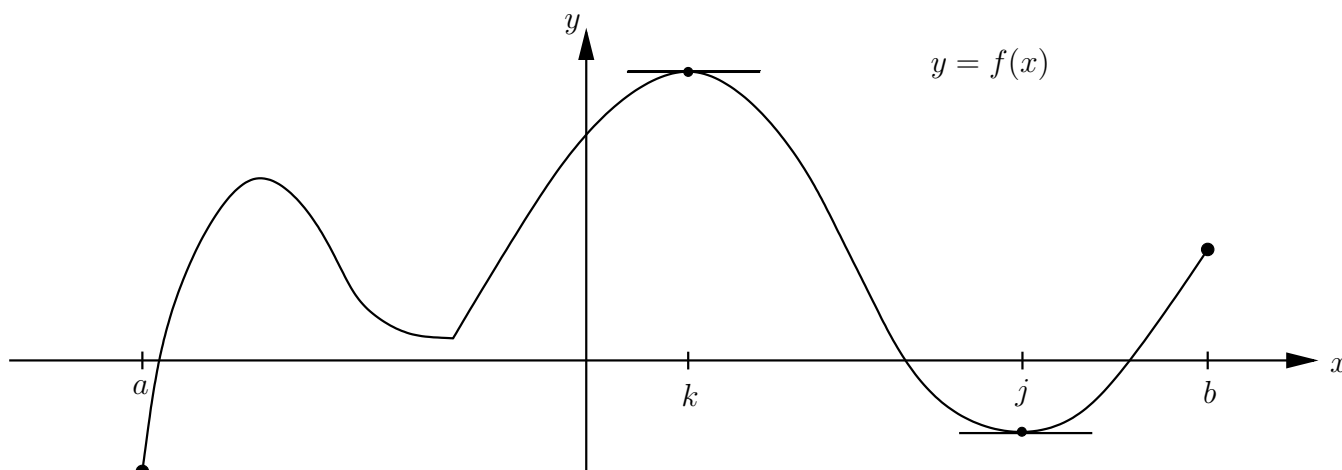
- (i) If $f''(x) > 0$ on an interval, then the graph of $y = f(x)$ is concave up on the interval.
- (ii) If $f''(x) < 0$ on an interval, then the graph of $y = f(x)$ is concave down on the interval.

Observe on our graph: whenever the graph of $y = f(x)$ changes from concave up to concave down, or vice versa, $f'(x)$ changes from increasing to decreasing, or vice versa. That is, $f''(x)$ changes from positive to negative, or vice versa. This may occur at points where $f''(x) = 0$ or $f''(x)$ does not exist.

Bottom Line: To determine the intervals of concavity of a function f :

- (i) Find points at which f'' changes sign (from positive to negative or vice versa). f'' can change sign at points where
 - $f''(x) = 0$
 - $f''(x)$ does not exist
 - $f(x)$ is not defined
- (ii) Test $f''(x)$ on the subintervals defined by the points from (i).

The second derivative can also be used to easily identify when a critical number corresponds to a relative minimum or maximum. Consider the relative maximum at $x = k$ and the relative minimum at $x = j$ shown on the following graph and consider f' and f'' at these two points:



The Second Derivative Test: Suppose $f''(x)$ is continuous near $x = c$.

- (i) If $f'(c) = 0$ and $f''(c) > 0$ then f has a relative minimum at $x = c$.
- (ii) If $f'(c) = 0$ and $f''(c) < 0$ then f has a relative maximum at $x = c$.

Example 1

Let $f(x) = x^4 - 2x^3 + 1$. Determine:

- (i) intervals of increase/decrease
- (ii) local extreme values
- (iii) intervals of concavity
- (iv) inflection points
- (v) Use (i)-(iv) to sketch the graph of $y = f(x)$.

Example 2

Determine the intervals of concavity and the inflection points of $f(x) = (x^2 - 4x + 2)e^x$.