Question 1:

(a)[3] Evaluate: $\lim\limits_{x\to\infty} \dfrac{7e^{5x} - 5e^{7x}}{7e^{7x} + 5e^{5x}}$

$$= \lim_{x\to\infty} \frac{\cancel{e^{7x}}(7e^{-2x}-5)}{\cancel{e^{7x}}(7+5e^{-2x})}$$

$$= \boxed{\frac{-5}{7}}$$

(b)[3] Differentiate: $f(x) = 5^{\sin(2x)+\sqrt{x}}$

$$f'(x) = 5^{\sin(2x)+\sqrt{x}} \cdot \ln 5 \cdot \left[2\cos(2x) + \tfrac{1}{2}x^{-\frac{1}{2}}\right]$$

(c)[4] Use logarithmic differentiation to determine y':

$$y = \frac{\cos^2(x)\sqrt{1+3x}}{e^{x^2}}$$

$$y = \frac{[\cos(x)]^2(1+3x)^{\frac{1}{2}}}{e^{x^2}}$$

$$\ln y = 2\ln(\cos x) + \tfrac{1}{2}\ln(1+3x) - x^2$$

$$\frac{1}{y}y' = \frac{-2\sin x}{\cos x} + \frac{3}{2(1+3x)} - 2x$$

$$\therefore y' = \frac{\cos^2(x)\sqrt{1+3x}}{e^{x^2}}\left[\frac{-2\sin x}{\cos x} + \frac{3}{2(1+3x)} - 2x\right]$$

Question 2:

(a)[3] Find y': $y = \log_5\left(xe^{-x}\right)$.

$$\boxed{y' = \frac{1}{xe^{-x}\cdot \ln 5}\cdot\left[e^{-x} - xe^{-x}\right]}$$

(b)[3] Determine $f''(1)$: $f(x) = x\,[1 - \ln(x)]$.

$$f'(x) = \cancel{1} - \ln(x) + \cancel{x\left(-\frac{1}{x}\right)}$$

$$= -\ln(x)$$

$$f''(x) = -\frac{1}{x}$$

$$f''(1) = -\frac{1}{1} = \boxed{-1}$$

(c)[4] Determine the equation of the tangent line to the curve

$$\ln(x + 2y) = e^{xy} - 1$$

at the point $(1, 0)$.

$$\frac{d}{dx}\left[\ln(x+2y)\right] = \frac{d}{dx}\left[e^{xy} - 1\right]$$

$$\frac{1}{x+2y}(1 + 2y') = e^{xy}(y + xy')$$

at $x = 1$, $y = 0$:

$$\cancel{\frac{1}{1+2\cdot 0}}^{\,1}(1 + 2y') = \cancel{e^{(1)(0)}}^{\,1}(0 + (1)y')$$

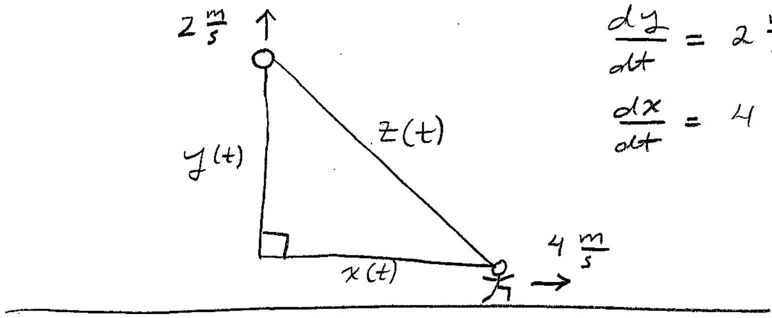
$$1 + 2y' = y'$$

$$\therefore\ y' = -1$$

$\therefore$ equation is $y - 0 = -1(x - 1)$

$$\boxed{y = -x + 1}$$

Question 3 [10]: A balloon is rising at a constant speed of 2 m/s. A person is running in a straight line on level ground at a constant speed of 4 m/s. At the instant when the person passes under the rising balloon it is 20 m above the person. How fast is the distance between the person and balloon increasing 5 seconds later? (Use calculus to find the solution, state a clear conclusion, and give units with your final answer.)



$$\frac{dy}{dt} = 2\,\frac{m}{s}$$

$$\frac{dx}{dt} = 4\,\frac{m}{s}.$$

5 seconds after person passes under the balloon:

$$y = 20\text{ m} + (2\,\tfrac{m}{s})(5\,s) = 30\text{ m}.$$

$$x = (4\,\tfrac{m}{s})(5\,s) = 20\text{ m}.$$

$\therefore$ Find $\frac{dz}{dt}$ when $x = 20$ m & $y = 30$ m.

$$z = [x^2+y^2]^{\frac{1}{2}}$$

$$\frac{dz}{dt} = \frac{1}{\cancel{2}}[x^2+y^2]^{-\frac{1}{2}}\left[\cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt}\right]$$

$$= \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{[x^2+y^2]^{\frac{1}{2}}}$$

When $x = 20$, $y = 30$:

$$\frac{dz}{dt} = \frac{(20)(4)+(30)(2)}{\sqrt{(20)^2+(30)^2}}$$

$$= \frac{140}{10\sqrt{13}}$$

$$= \frac{14}{\sqrt{13}}$$

$\therefore$ The distance is increasing by $\frac{14}{\sqrt{13}}\,\frac{m}{s}$.

Question 4:

(a)[5] Determine the linearization $L(x)$ of $f(x) = e^{x-(1/x)}$ at $a = 1$.

$f(x) = e^{x-\frac{1}{x}}$; $f(a) = f(1) = e^{1-\frac{1}{1}} = e^0 = 1$

$f'(x) = e^{x-\frac{1}{x}}\left(1+\frac{1}{x^2}\right)$; $f'(1) = e^{1-\frac{1}{1}}\left(1+\frac{1}{1^2}\right) = 2$

$\therefore$ $L(x) = f(a) + f'(a)(x-a)$

$\boxed{L(x) = 1 + 2(x-1)}$

(b)[5] Use a linear approximation (or differentials if you prefer) to estimate $\sqrt{9.1}$.

Here $f(x) = \sqrt{x}$, $a = 9$, $f(a) = 3$

$f'(x) = \frac{1}{2\sqrt{x}}$; $f'(a) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$.

$\therefore$ $L(x) = f(a) + f'(a)(x-a)$.

$= 3 + \frac{1}{6}(x-9)$

$\sqrt{9.1} \simeq L(9.1)$

$= 3 + \frac{1}{6}(9.1-9)$

$= 3 + \frac{1}{6}\left(\frac{1}{10}\right)$

$= \boxed{\frac{181}{60}}$

Question 5: For this question use the function $f(x) = (x^2 - 9)^2$. } Domain $(-\infty, \infty)$.

(a)[4] Determine the intervals of increase and decrease of $f(x)$.

$$f'(x) = 2(x^2-9)(2x)$$
$$= 4x(x-3)(x+3)$$

- $f'(x) = 0$ at $x = 0, 3, -3$
- $f'(x)$ exists for every x.

		-4	-3	-1	0	1	3	4
$f'(x) = 4x(x-3)(x+3)$:		$-$	0	$+$	0	$-$	0	$+$
$f(x) = (x^2-9)^2$:		$\searrow$	0	$\nearrow$	81	$\searrow$	0	$\nearrow$

$\therefore$ f is increasing on $(-3, 0) \cup (3, \infty)$; f is decreasing on $(-\infty, -3) \cup (0, 3)$.

(b)[1] State the relative extreme values of $f(x)$, if any.

f has a rel. min. of 0 at $x = -3$,
a rel. max. of 81 at $x = 0$,
a rel. min. of 0 at $x = 3$,

(c)[4] Determine the intervals of concavity of $f(x)$.

$$f'(x) = 4x(x^2-9)$$
$$f''(x) = 4(x^2-9) + 4x(2x)$$
$$= 12x^2 - 36$$
$$= 12(x^2-3).$$

- $f''(x) = 0$ at $x = \pm\sqrt{3}$
- $f''(x)$ exists for every x

	-2	$-\sqrt{3}$	0	$\sqrt{3}$	2
$f''(x) = 12(x^2-3)$:	$+$	0	$-$	0	$+$
$f(x) = (x^2-9)^2$:	$\smile$	36	$\frown$	36	$\smile$

$\therefore$ Graph of $y = f(x)$ is concave up on $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$, concave down on $(-\sqrt{3}, \sqrt{3})$.

(d)[1] State the inflections points, if any.

Graph of $y = f(x)$ has inflection points at $(-\sqrt{3}, 36)$ and $(\sqrt{3}, 36)$.