Question 1: Evaluate the following limits, showing all work. If a limit does not exist but is ∞ or $-\infty$, state which, with an explanation of your reasoning.

(a)[2] $\displaystyle\lim_{x\to 7^-} \frac{7x}{x-7}$

As $x \to 7^-$, $7x \to 49$, while $x-7 \to 0^-$

$\therefore \displaystyle\lim_{x\to 7^-} \frac{7x}{x-7} = \boxed{-\infty}$

(b)[2] $\displaystyle\lim_{t\to -\infty} \frac{t^3-t+1}{t^4+3t^2-2} = \lim_{t\to\infty} \frac{t^3\left(1-\frac{1}{t^2}+\frac{1}{t^3}\right)}{t^4\left(1+\frac{3}{t^2}-\frac{2}{t^4}\right)}$

$\displaystyle = \lim_{t\to\infty} \frac{1-\frac{1}{t^2}+\frac{1}{t^3}}{t\left(1+\frac{3}{t^2}-\frac{2}{t^4}\right)}$

$= \boxed{0}$

(c)[3] $\displaystyle\lim_{x\to\infty} x^2-x^4 = \lim_{t\to\infty} x^4\left(\frac{1}{x^2}-1\right)$

$= \boxed{-\infty}$ since $x^4 \to \infty$ while $\frac{1}{x^2}-1 \to -1$

(d)[3] $\displaystyle\lim_{x\to\infty} \frac{x+2}{\sqrt{9x^2+1}} = \lim_{x\to\infty} \frac{\cancel{x}\left(1+\frac{2}{x}\right)}{3\cancel{x}\sqrt{1+\frac{1}{9x^2}}} = \boxed{\frac{1}{3}}$

Question 2: Suppose $h(t) = 20t - kt^2$ describes the height in metres above the ground of a projectile launched with an initial velocity of 20 m/s, where time $t \geq 0$. Here k is a positive constant, and note that some of your answers may contain this constant k. State units with your answers.

(a)[2] At what time(s) is the object at ground level?

When $h(t) = 0$: $20t - kt^2 = 0$

$$t(20 - kt) = 0$$

$$t = 0, \quad 20 - kt = 0$$

$$\boxed{\therefore\ t = 0 \text{ sec},\ t = \frac{20}{k} \text{ sec}}$$

(b)[3] At what time in the projectile's flight will it begin falling back to earth?

When $h'(t) = 0$: $h'(t) = \frac{d}{dt}\left[20t - kt^2\right]$

$$= 20 - 2kt$$

$$h'(t) = 0 \Rightarrow 20 - 2kt = 0$$

$$\therefore\ t = \frac{20}{2k} = \boxed{\frac{10}{k} \text{ sec}}$$

(c)[2] What is the acceleration of the projectile at time $t = 1$ second?

$$h'(t) = 20 - 2kt$$

$$h''(t) = -2k$$

$$h''(1) = \boxed{-2k\ \frac{m}{s^2}}$$

(d)[3] What is the velocity of the projectile when it hits the ground?

Find $h'\left(\frac{20}{k}\right) = 20 - 2k\left(\frac{20}{k}\right)$

$$= \boxed{-20\ \frac{m}{s}}$$

Question 3: Determine the derivatives of the following functions. It is not necessary to simplify your final answers.

(a)[3] $y = \dfrac{x^3}{3} - 3\sqrt[3]{x} + \sqrt{3} = \frac{1}{3}x^3 - 3x^{1/3} + \sqrt{3}$

$$\therefore\ y' = \tfrac{1}{3}(3x^2) - 3\left(\tfrac{1}{3}x^{-2/3}\right) + 0$$

$$= \boxed{x^2 - x^{-2/3}}$$

(b)[4] $f(x) = (x - \cos x)\left(\dfrac{4}{x} - \sin x\right) = (x - \cos x)(4x^{-1} - \sin x)$

$$\boxed{f'(x) = (1 + \sin x)(4x^{-1} - \sin x) + (x - \cos x)(-4x^{-2} - \cos x)}$$

(c)[3] $g(x) = \dfrac{\tan(2x)}{\pi^2 + x^2}$

$$\boxed{g'(x) = \frac{(\pi^2 + x^2)\sec^2(2x)\cdot 2 - \tan(2x)(2x)}{(\pi^2 + x^2)^2}}$$

Question 4: Determine the derivatives of the following functions. It is not necessary to simplify your final answers.

(a)[3] $y = \sec\left(\sqrt{x} - x^5\right) = \sec\left(x^{\frac{1}{2}} - x^5\right)$

$$\boxed{y' = \sec\left(x^{\frac{1}{2}} - x^5\right)\tan\left(x^{\frac{1}{2}} - x^5\right)\left(\frac{1}{2}x^{-\frac{1}{2}} - 5x^4\right)}$$

(b)[3] $f(\theta) = \sqrt{3\theta - \theta\sin\theta} = \left(3\theta - \theta\sin\theta\right)^{\frac{1}{2}}$

$$\boxed{f'(\theta) = \frac{1}{2}\left(3\theta - \theta\sin\theta\right)^{-\frac{1}{2}}\left(3 - \sin\theta - \theta\cos\theta\right)}$$

(c)[4] $g(t) = \left[t + \cos\left(\frac{1}{\sqrt{t}}\right)\right]^{121} = \left[t + \cos\left(t^{-\frac{1}{2}}\right)\right]^{121}$

$$\boxed{g'(t) = 121\left[t + \cos\left(t^{-\frac{1}{2}}\right)\right]^{120}\left[1 - \sin\left(t^{-\frac{1}{2}}\right)\left(-\frac{1}{2}t^{-\frac{3}{2}}\right)\right]}$$

Question 5:

(a)[5] Determine the equation of the tangent line to the curve $x^3 - 5xy^2 + y^3 = xy - 3$ at the point $(2, 1)$.

$$\frac{d}{dx}\left[x^3 - 5xy^2 + y^3\right] = \frac{d}{dx}\left[xy - 3\right]$$

$$3x^2 - 5y^2 - 5x2yy' + 3y^2y' = y + xy'$$

at $x=2$, $y=1$:

$$3(2)^2 - 5(1)^2 - 5(2)(2)(1)y' + 3(1)^2y' = 1 + (2)y'$$

$$7 - 17y' = 1 + 2y'$$

$$y' = \frac{6}{19}$$

$\therefore$ Equation is $\boxed{y - 1 = \frac{6}{19}(x-2)}$

(b)[5] There are two tangent lines to the curve $y = x^2 + x$ which pass through the point $(2, -3)$. Determine the points at which these tangent lines contact the curve.

Let (a, a^2+a) be a point of intersection of the tangent line with the curve.

Then tangent line has slope $m = \frac{a^2+a-(-3)}{a-2} = \frac{a^2+a+3}{a-2}$.

But also, $m = \frac{d}{dx}\left[x^2+x\right]\Big|_{x=a} = 2x+1\Big|_{x=a} = 2a+1$

$$\therefore \frac{a^2+a+3}{a-2} = 2a+1$$

$$a^2+a+3 = (2a+1)(a-2)$$

$$a^2+a+3 = 2a^2-3a-2$$

$$a^2-4a-5=0$$

$$(a-5)(a+1)=0$$

$\therefore$ $a=5$, $a^2+a=30$;

$a=-1$, $a^2+a=0$

$\boxed{\therefore \text{ Points are } (5,30) \ \& \ (-1,0)}$