Question 1 [12 points]: Evaluate the following limits, if they exist. If a limit does not exist because it is $\pm\infty$, state which it is and include an explanation of your reasoning. You may use the fact that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

(a)[3] $\lim_{x\to 2} \frac{x^2-4}{x^2-5x+6} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}(x-3)} = \boxed{-4}$

(b)[3] $\lim_{x\to -\infty} e^{\cos(1/x)}$

as $x \to -\infty$, $\frac{1}{x} \to 0$,

So $\cos\left(\frac{1}{x}\right) \to 1$, so

$\lim_{x\to -\infty} e^{\cos\left(\frac{1}{x}\right)} = \boxed{e}$

(c)[3] $\lim_{x\to 9} \frac{3-\sqrt{x}}{9-x} \cdot \frac{3+\sqrt{x}}{3+\sqrt{x}} = \lim_{x\to 9} \frac{\cancel{(9-x)}}{\cancel{(9-x)}(3+\sqrt{x})} = \boxed{\frac{1}{6}}$

(d)[3] $\lim_{x\to 0} \frac{\sin(\pi x)}{\pi \sin(3x)} = \lim_{x\to 0} \frac{1}{\pi}\left[\frac{\frac{\sin(\pi x)}{\pi x} \cdot \pi \cancel{x}}{\frac{\sin(3x)}{3x} \cdot 3\cancel{x}}\right]$

$= \frac{1}{\cancel{\pi}}\left[\frac{1 \cdot \cancel{\pi}}{1 \cdot 3}\right]$

$= \boxed{\frac{1}{3}}$

Question 2 [15 points]: Differentiate the following functions (you do not have to simplify your answers):

(a)[3] $y = 4x^3 - \sqrt{x} + \tan x - \pi \quad = 4x^3 - x^{1/2} + \tan x - \pi$

$$\boxed{y' = 12x^2 - \tfrac{1}{2}x^{-\frac{1}{2}} + \sec^2 x}$$

(b)[3] $y = \sqrt{3 + \cos^2 x} \quad = \left[3 + (\cos x)^2\right]^{1/2}$

$$\boxed{y' = \tfrac{1}{2}\left[3 + \cos^2 x\right]^{-1/2}\left[-2\cos x \sin x\right]}$$

(c)[3] $y = \dfrac{t - \sqrt{t}}{e^t} \quad = \dfrac{t - t^{1/2}}{e^t}$

$$\boxed{y' = \frac{e^t\left[1 - \frac{1}{2}t^{-1/2}\right] - \left[t - t^{1/2}\right]e^t}{e^{2t}}}$$

(d)[3] $y = \sec(x)\ln(1+x)$

$$\boxed{y' = \sec(x)\tan(x)\ln(1+x) + \sec(x)\frac{1}{1+x}}$$

(e)[3] $y = \sin\left(2^{x^2+x} - \log_2 x\right)$

$$\boxed{y' = \cos\left(2^{x^2+x} - \log_2 x\right) \cdot \left[2^{x^2+x}\ln(2)\cdot(2x+1) - \frac{1}{x\ln(2)}\right]}$$

Question 3 [10 points]:

(a)[2] Find the general antiderivative of $f(x) = 2x^{1/2} + 3x^{1/3}$

$$F(x) = 2\frac{x^{3/2}}{(3/2)} + 3\frac{x^{4/3}}{(4/3)} + C$$

$$\boxed{F(x) = \frac{4}{3}x^{3/2} + \frac{9}{4}x^{4/3} + C}$$

(b)[2] Find the general antiderivative of $g(x) = \csc^2 x - 5\cos x - \pi$

$$\boxed{G(x) = -\cot(x) - 5\sin(x) - \pi x + C}$$

(c)[2] Find the general antiderivative of $f(x) = \dfrac{x^3 - 7x + \sqrt{x}}{x^2} = x - 7\left(\frac{1}{x}\right) + x^{-3/2}$

$$F(x) = \frac{x^2}{2} - 7\ln|x| + \frac{x^{-1/2}}{(-1/2)} + C$$

$$\boxed{F(x) = \frac{x^2}{2} - 7\ln|x| - 2x^{-1/2} + C}$$

(d)[4] Find the function $g(t)$ such that $g''(t) = 6t - \sin t$ and $g(0) = 1$, $g'(0) = 1$.

$$g'(t) = \frac{6t^2}{2} + \cos(t) + C_1$$

$$g'(0) = 1 \Rightarrow \cancel{3 \cdot 0^2}^{\,0} + \cancel{\cos(0)}^{\,1} + C_1 = 1$$

$$\therefore C_1 = 0.$$

$$\therefore g'(t) = 3t^2 + \cos(t)$$

$$\therefore g(t) = \frac{3t^3}{3} + \sin(t) + C_2$$

$$g(0) = 1 \Rightarrow \cancel{0^3}^{\,0} + \cancel{\sin(0)}^{\,0} + C_2 = 1$$

$$\therefore C_2 = 1$$

$$\boxed{\therefore g(t) = t^3 + \sin(t) + 1}$$

Question 4 [8 points]: Use the <u>**definition of the derivative**</u> to find $f'(x)$ for $f(x) = \frac{2}{x^2}$. (A score of 0 will be given if $f'(x)$ is found using the differentiation rules, though you may check your answer using the rules.)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{2}{(x+h)^2} - \frac{2}{x^2} \right]$$

$$= \lim_{h \to 0} \frac{2}{h} \left[\frac{x^2 - (x+h)^2}{(x+h)^2 x^2} \right]$$

$$= \lim_{h \to 0} \frac{2}{h} \left[\frac{\cancel{x^2} - \cancel{x^2} - 2xh - h^2}{(x+h)^2 x^2} \right]$$

$$= \lim_{h \to 0} \frac{2}{\cancel{h}} \cdot \frac{\cancel{h}(-2x - h)}{(x+h)^2 x^2}$$

$$= \frac{-4x}{x^2 \cdot x^2}$$

$$= \boxed{\frac{-4}{x^3}}$$

Question 5 [10 points]:

(a)[5] Use implicit differentiation to find an equation of the tangent line to the curve

$$x\sin(x-y^2) = x^2 - 1$$

at the point $(1,1)$.

$$\frac{d}{dx}\left[x\sin(x-y^2)\right] = \frac{d}{dx}\left[x^2-1\right]$$

$$\sin(x-y^2) + x\cos(x-y^2)(1-2yy') = 2x$$

at $(1,1)$:

$$\underbrace{\cancel{\sin(1-1^2)}}_{\to 0} + 1\cdot \underbrace{\cancel{\cos(1-1^2)}}_{\to 1}(1-2\cdot 1\cdot y') = 2\cdot 1$$

$$1-2y' = 2$$

$$y' = \frac{2-1}{-2}$$

$$y' = -\frac{1}{2}$$

$\therefore$ Equation of tangent line is

$$\boxed{y-1 = -\frac{1}{2}(x-1).}$$

(b)[5] Use logarithmic differentiation to find $\frac{dy}{dx}$ for $y = \frac{5^x(x^2-1)}{(x^3+1)^7}$.

$$\ln y = \ln\left[\frac{5^x(x^2-1)}{(x^3+1)^7}\right]$$

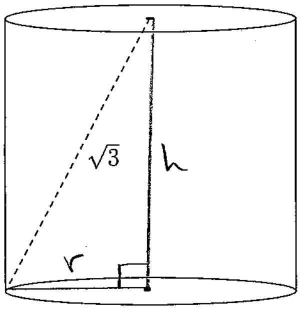
$$\ln y = x\ln 5 + \ln(x^2-1) - 7\ln(x^3+1)$$

$$\frac{d}{dx}\left[\ln y\right] = \frac{d}{dx}\left[x\ln 5 + \ln(x^2-1) - 7\ln(x^3+1)\right]$$

$$\frac{1}{y}y' = \ln 5 + \frac{2x}{x^2-1} - 7\cdot\frac{3x^2}{x^3+1}$$

$$\boxed{y' = \frac{5^x(x^2-1)}{(x^3+1)^7}\left[\ln 5 + \frac{2x}{x^2-1} - \frac{21x^2}{x^3+1}\right]}$$

Question 6 [10 points]: The distance from the lower edge to the centre of the top of a closed cylinder is $\sqrt{3}$ m as shown in the figure below. Determine the maximum volume of such a cylinder, taking care to justify using one of the appropriate tests that your result does indeed correspond to the maximum. (Recall that the volume of a cylinder of base radius r and height h is $V = \pi r^2 h$.)



$V = \pi r^2 h$

$r^2 + h^2 = (\sqrt{3})^2 = 3$

$\therefore r^2 = 3 - h^2$

$\therefore V = \pi (3 - h^2) h$

$= \pi [3h - h^3]$.

Maximize $V(h) = \underbrace{\pi [3h - h^3]}_{\text{continuous}}$, $\underbrace{0 \leq h \leq \sqrt{3}}_{\text{closed interval}}$

$V'(h) = \pi [3 - 3h^2]$

$V'(h) = 0 \Rightarrow \pi [3 - 3h^2] = 0$

$h^2 = 1$

$h = 1$

h	$V(h) = \pi [3h - h^3]$
0	0
1	2π
$\sqrt{3}$	0

$\therefore$ The maximum volume is $2\pi \, m^3$.

Question 7 [15 points]: For this question let $f(x) = \dfrac{x^2+12}{2x+1}$. Note that the graph of $y = f(x)$ has y-intercept $(0, 12)$, and that

$$\lim_{x\to\infty} f(x) = \infty, \qquad \lim_{x\to-\infty} f(x) = -\infty$$

(a)[1] Determine the domain of $f(x)$.

Domain of f is $(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, \infty)$.

(b)[4] Determine the intervals of increase and decrease of $f(x)$.

$$f'(x) = \frac{(2x+1)(2x) - (x^2+12)(2)}{(2x+1)^2}$$

$$= \frac{4x^2+2x-2x^2-24}{(2x+1)^2}$$

$$= \frac{2(x^2+x-12)}{(2x+1)^2}$$

$$= \frac{2(x-3)(x+4)}{(2x+1)^2}$$

- $f'(x) = 0$: $x = 3$, $x = -4$
- $f'(x)$ does not exist : $x = \frac{-1}{2}$

		-4		$-\frac{1}{2}$		3	
test :	-5		-1		0		4
$f'(x) = \frac{2(x-3)(x+4)}{(2x+1)^2}$:	$+$	0	$-$	NA	$-$	0	$+$
$f(x) = \frac{x^2+12}{2x+1}$:	$\nearrow$	-4	$\searrow$	NA	$\searrow$	3	$\nearrow$

$\therefore$ f is increasing on $(-\infty, -4)$ and $(3, \infty)$;
f is decreasing on $(-4, -\frac{1}{2})$ and $(\frac{-1}{2}, 3)$.

(c)[1] State the relative extrema of $f(x)$.

f has a rel. max. of -4 at $x = -4$;
f has a rel. min. of 3 at $x = 3$.

(continued on next page...)

(d)[3] Determine the intervals of concavity of $f(x)$. You may use the fact that $f''(x) = \dfrac{98}{(2x+1)^3}$.

- $f''(x) = 0$? no such x
- $f''(x)$ not exist ? $x = \frac{-1}{2}$

	-1	$-\frac{1}{2}$	0
test:			
$f''(x) = \dfrac{98}{(2x+1)^3}$:	$-$	NA	$+$
$f(x)$:	$\frown$	NA	$\smile$

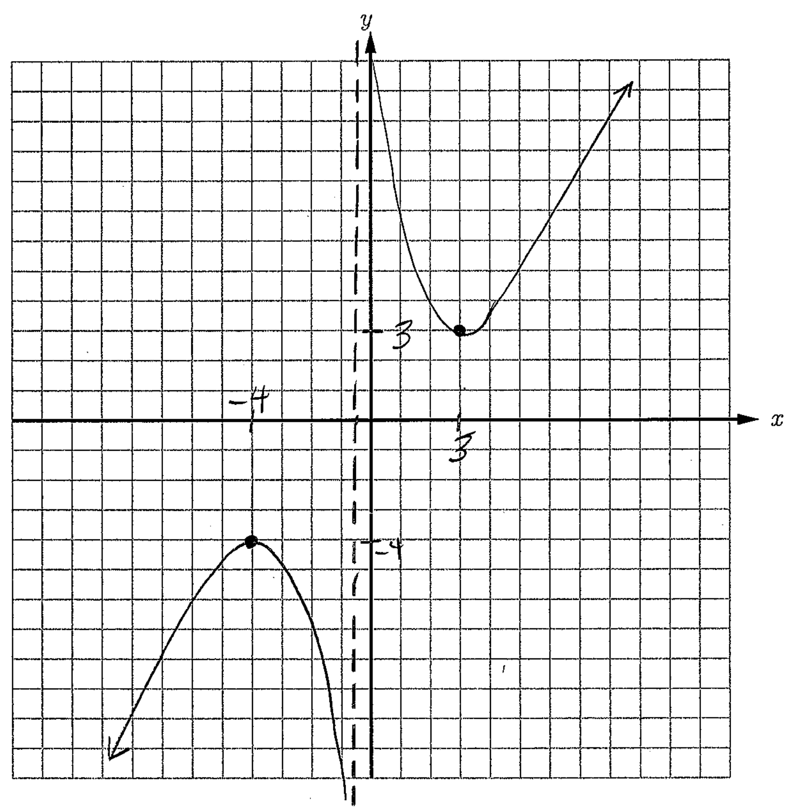
$\therefore$ f is concave down on $(-\infty, \frac{-1}{2})$;
f is concave up on $(\frac{-1}{2}, \infty)$

(e)[2] Determine the vertical asymptotes, if any.

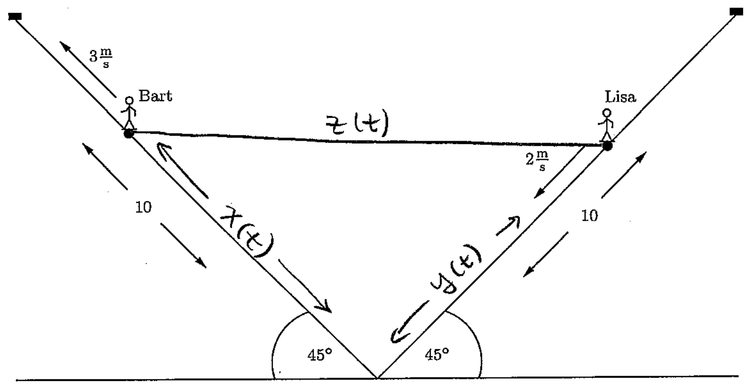
$$\left.\begin{aligned} \lim_{x \to -\frac{1}{2}^+} \frac{x^2+12}{2x+1} &= +\infty \\ \lim_{x \to -\frac{1}{2}^-} \frac{x^2+12}{2x+1} &= -\infty \end{aligned}\right\}$$

$\therefore$ $x = \frac{-1}{2}$ is a vertical asymptote.

(f)[4] Sketch the graph of $y = f(x)$.



Question 8 [10 points]: Bart steps onto the bottom of a 10 m long escalator moving at 3 m/s. At the same instant Lisa steps onto the top of a 10 m long escalator moving at 2 m/s. Each escalator makes an angle of 45° with the ground, and the bottoms of the escalators meet at the same point at ground level as shown in the figure below. Use calculus to determine the rate at which the distance between Bart and Lisa is changing two seconds after they step onto their respective escalators.



$\frac{dx}{dt} = 3 \frac{m}{s}$; $\frac{dy}{dt} = -2 \frac{m}{s}$.

At $t = 2\,s$: $x = (3 \frac{m}{s})(2\,s) = 6$ m

$y = 10 - (2\frac{m}{s})(2\,s) = 6$ m.

$z = (x^2+y^2)^{\frac{1}{2}}$

Find $\frac{dz}{dt}$ when $x=6$, $y=6$.

$$\frac{dz}{dt} = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right)$$

When $x=6$, $y=6$:

$$\frac{dz}{dt} = \frac{1}{2\sqrt{6^2+6^2}}\left[(2)(6)(3) + (2)(6)(-2)\right]$$

$$= \frac{12}{2\sqrt{6^2 \cdot 2}}$$

$$= \frac{1}{\sqrt{2}}$$

$\therefore$ distance is increasing by $\frac{1}{\sqrt{2}}\ \frac{m}{s}$.

Question 9 [10 points]:

(a)[5] Let $f(x) = \dfrac{1}{e^x + 1}$. Use a linear approximation to estimate $f(0.1)$.

$f(x) = \dfrac{1}{e^x+1}$, $a = 0.$

$f(a) = f(0) = \dfrac{1}{e^0+1} = \dfrac{1}{2}$

$f'(x) = \dfrac{-1}{(e^x+1)^2} \cdot e^x$

$f'(a) = f'(0) = \dfrac{-1}{(e^0+1)^2} \cdot e^0 = \dfrac{-1}{4}$

$\therefore$ $L(x) = f(a) + f'(a)(x-a)$

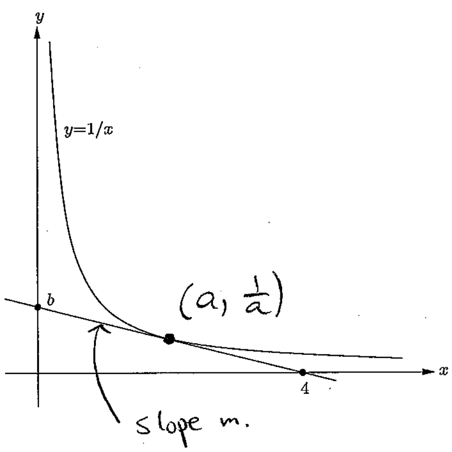
$= \dfrac{1}{2} - \dfrac{1}{4}x$

$\therefore$ $f(0.1) \approx L\left(\dfrac{1}{10}\right) = \dfrac{1}{2} - \dfrac{1}{4}\left(\dfrac{1}{10}\right)$

$= \dfrac{20-1}{40}$

$= \boxed{\dfrac{19}{40}}$

(b)[5] In the figure below the line is tangent to the given curve. Determine the value of b.



$f'(x) = \dfrac{-1}{x^2}$

$f'(a) = \dfrac{-1}{a^2}$

also, $m = \dfrac{\frac{1}{a} - 0}{a-4}$

$\therefore$ $\dfrac{-1}{a^2} = \dfrac{\left(\frac{1}{a}\right)}{a-4}$

$4 - a = a$

$4 = 2a$

$a = 2$

$\therefore$ $m = f'(a) = \dfrac{-1}{2^2} = \dfrac{-1}{4}$

$\therefore$ $\dfrac{-b}{4} = \dfrac{-1}{4}$

$\boxed{b=1}$.

Question 10 [5 points]: Let $f(x) = e^{2x}$ and $g(x) = k + xe^{2x}$ where k is some constant. The graphs of $y = f(x)$ and $y = g(x)$ intersect at a single point, and both curves have the same tangent line at the point of intersection. Determine the value of k.

Let $x = a$ be the x coordinate of the point of intersection.

$\therefore \quad f'(a) = g'(a)$

$\therefore \quad 2e^{2x}\Big|_{x=a} = e^{2x} + 2xe^{2x}\Big|_{x=a}$

$2e^{2a} = e^{2a} + 2ae^{2a}$

$2 = 1 + 2a$

$a = \frac{2-1}{2} = \frac{1}{2}$

At $x = a = \frac{1}{2}$: $\quad f(a) = g(a)$

$e^{2(\frac{1}{2})} = k + \frac{1}{2}e^{2(\frac{1}{2})}$

$e = k + \frac{1}{2}e$

$\therefore \boxed{k = \frac{e}{2}}$