Question 1:

(a)[3 points] Find the linearization $L(x)$ (that is, the linear approximation) of $g(x) = \cos^2 x$ at $a = \pi/4$.

$$g(\tfrac{\pi}{4}) = \cos^2(\tfrac{\pi}{4}) = \left(\tfrac{1}{\sqrt{2}}\right)^2 = \tfrac{1}{2}$$

$$g'(x) = -2\cos(x)\sin(x) \;;\; g'(\tfrac{\pi}{4}) = -2\cos(\tfrac{\pi}{4})\sin(\tfrac{\pi}{4}) = -2\left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{1}{\sqrt{2}}\right) = -1$$

$$\therefore L(x) = g(\tfrac{\pi}{4}) + g'(\tfrac{\pi}{4})(x - \tfrac{\pi}{4})$$

i.e. $\boxed{L(x) = \tfrac{1}{2} - (x - \tfrac{\pi}{4})}$

(b)[3 points] The point $(4, -1)$ is on the graph of the function $f(x)$. Using a linear approximation (or differentials), $f(3.9) \approx -0.8$. Determine the slope of the tangent line to $y = f(x)$ at the point where $x = 4$.

The points $(4,-1)$ and $(3.9,-0.8)$ are both on the tangent line, so the slope of the tangent line is $m = \dfrac{-1-(-0.8)}{4-3.9} = \dfrac{-0.2}{0.1} = \boxed{-2}$

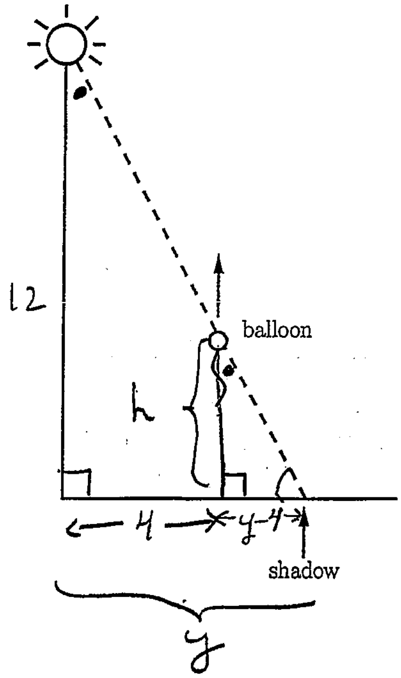
(c)[4 points] Compute $q''(0)$ if $q(x) = \log_2(2^x + 1)$.

$$q'(x) = \frac{1}{(2^x+1)\cancel{\ln 2}} \cdot \left(2^x \cancel{\ln 2}\right) = \frac{2^x}{2^x+1}$$

$$q''(x) = \left[\frac{(2^x+1)2^x\ln 2 - 2^x 2^x \ln 2}{(2^x+1)^2}\right]$$

$$q''(0) = \left[\frac{(2^0+1)\cdot 2^0 \cdot \ln 2 - 2^0 2^0 \ln 2}{(2^0+1)^2}\right] = \left[\frac{2\ln 2 - \ln 2}{4}\right] = \boxed{\frac{(\ln 2)}{4}}$$

Question 2 [10 points]: A balloon is released from ground level 4 m from the base of a 12 m tall lamppost. As the balloon rises vertically at a rate of 2 m/s it casts a shadow on the ground as a result of the light atop the lamppost. As what rate is the shadow moving along the ground 2 s after the balloon is released?



$$\frac{dh}{dt} = 2\,\frac{m}{s}$$

Find $\frac{dy}{dt}$ when $h = (2\,s)(2\,\frac{m}{s}) = 4m.$

By similar triangles,

$$\frac{y-4}{h} = \frac{y}{12}$$

$$\therefore\ 12y - 48 = yh$$

$$y(12-h) = 48$$

$$y = \frac{48}{12-h}$$

$$\therefore\ \frac{dy}{dt} = \frac{-48}{(12-h)^2}\cdot\left(-\frac{dh}{dt}\right)$$

when $h = 4\,m$: $\frac{dy}{dt} = \frac{48}{(12-4)^2}\cdot 2 = \frac{3}{2}\,\frac{m}{s}.$

$\therefore$ The shadow is moving along the ground at $\frac{3}{2}\,\frac{m}{s}$.

Question 3:

(a)[5 points] Determine the equation of the tangent line to the curve $xe^y = y - 1$ at the point where $x = 0$.

at $x=0$: $\cancel{0 \cdot e^{y}} \rightarrow 0 = y-1 \Rightarrow y=1$

$\frac{d}{dx}[xe^y] = \frac{d}{dx}[y-1]$

$e^y + xe^y y' = y'$

At $x=0, y=1$:

$e + 0 = y'$

$\therefore y' = e.$

$\therefore$ Equation of tangent line is $y-1 = e(x-0)$

or $\boxed{y = ex+1}$

(b)[5 points] Use logarithmic differentiation to determine y': $y = xe^{\sin^2 x}$

$\ln y = \ln(xe^{\sin^2 x})$

$\ln y = \ln x + \sin^2 x$

$\frac{1}{y} y' = \frac{1}{x} + 2\sin x \cos x$

$\therefore \boxed{y' = xe^{\sin^2 x}\left[\frac{1}{x} + 2\sin x \cos x\right]}$

Question 4 [10 points]: Determine the absolute maximum and absolute minimum values of $\underbrace{f(x) = x(2 - \ln x)}_{\text{continuous}}$ on the interval $\underbrace{[1, e^2]}_{\text{closed}}$.

$$f'(x) = 2 - \ln x + x\left(\frac{-1}{x}\right)$$
$$= 2 - \ln x - 1$$
$$= 1 - \ln x$$

• $f'(x) = 0$? $\quad 1 - \ln x = 0$
$$\ln x = 1$$
$$x = e.$$

• $f'(x)$ not exist ? no such x

$\therefore$ critical number is $x = e$.

Test:

x	$f(x) = x(2-\ln x)$
1	$1\cdot(2-\ln 1) = 2$
e	$e\cdot(2-\ln e) = e$
e^2	$e^2(2-\ln e^2) = 0$

$\therefore$ f has an abs. max. of e at $x = e$

f has an abs. min. of 0 at $x = e^2$

Question 5:

(a) [8 points] Determine the intervals of increase and decrease of $f(x) = (x^2 - 4)^{2/3}$.

$$f'(x) = \frac{2}{3}(x^2-4)^{-\frac{1}{3}}(2x) = \frac{4x}{3(x^2-4)^{1/3}}$$

note that the domain of f is all real x.

- $f'(x) = 0$? $\quad \dfrac{4x}{3(x^2-4)^{1/3}} = 0 \Rightarrow x = 0$
- $f'(x)$ not exist ? $\quad x^2-4 = 0 \Rightarrow x = -2, 2.$

test :	-3	-2	-1	0	1	2	3
$f'(x) = \dfrac{4x}{3(x^2-4)^{1/3}}$:	$-$	NA	$+$	0	$-$	NA	$+$
$f(x) = (x^2-4)^{2/3}$:	$\searrow$	0	$\nearrow$	$4^{2/3}$	$\searrow$	0	$\nearrow$

$\therefore$ f is decreasing on $(-\infty, -2)$ and $(0, 2)$

f is increasing on $(-2, 0)$ and $(2, \infty)$.

(b) [2 points] Use your result from part (a) to determine the relative extrema of $f(x)$.

f has a rel. min. of 0 at $x = -2$,

a rel. min of 0 at $x = 2$

a rel. max. of $4^{2/3}$ at $x = 0$.