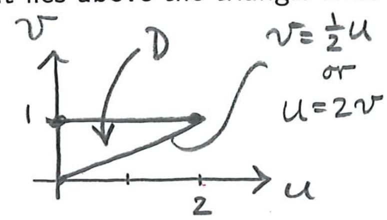
Question 1: Calculate the area of the part of the surface $z = 1 + 3x + 2y^2$ that lies above the triangle with vertices $(0,0)$, $(0,1)$ and $(2,1)$.



Surface parametrization : $\vec{r} = \langle u, v, 1+3u+2v^2 \rangle$

$$|\vec{r}_u \times \vec{r}_v| = \left| \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 3 \\ 0 & 1 & 4v \end{bmatrix} \right|$$

$$= | \langle -3, -4v, 1 \rangle |$$

$$= \sqrt{10 + 16v^2}$$

$$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA$$

$$= \int_{u=0}^{2} \int_{v=\frac{1}{2}u}^{1} \sqrt{10+16v^2} \, dv \, du$$ } hard: reverse order of integration

$$= \int_{v=0}^{1} \int_{u=0}^{2v} (10+16v^2)^{\frac{1}{2}} \, du \, dv$$

$$= \int_{v=0}^{1} (10+16v^2)^{\frac{1}{2}} \, 2v \, dv$$ } let $w = 10 + 16v^2$, $dw = 32v\,dv$

$$= \left(\frac{1}{16}\right)\left(\frac{2}{3}\right) (10+16v^2)^{3/2} \Big|_0^1$$

$$= \boxed{\frac{1}{24}\left(26^{3/2} - 10^{3/2}\right)}$$

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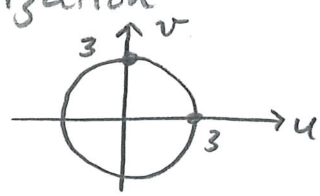
Question 2: Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = x^2 z\,\mathbf{i} + xy^2\,\mathbf{j} + z^2\,\mathbf{k}$ and C is the curve of intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 9$ oriented counterclockwise if viewed from above. (Hint: Stokes Theorem.)

By Stokes Thm, $I = \int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$

The enclosed surface has parametrization

$\vec{r} = \langle u, v, 1-u-v \rangle$ where D :



and normal to surface is

$$\vec{r_u} \times \vec{r_v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \langle 1, 1, 1 \rangle.$$

$$\text{curl}(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 z & xy^2 & z^2 \end{vmatrix} = \langle 0, x^2, y^2 \rangle = \langle 0, u^2, v^2 \rangle$$

$$\therefore I = \iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$$

$$= \iint_D \text{curl}(\vec{F}) \cdot (\vec{r_u} \times \vec{r_v})\, dA$$

$$= \iint_D \langle 0, u^2, v^2 \rangle \cdot \langle 1, 1, 1 \rangle\, dA$$

$$= \iint_D (u^2 + v^2)\, dA$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{3} r^2\, r\, dr\, d\theta$$

$$= 2\pi \left[\frac{r^4}{4} \right]_0^3 = \boxed{\frac{81\pi}{2}}$$

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Question 3: Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = x^4\,\mathbf{i} - x^3z^2\,\mathbf{j} + 4xy^2z\,\mathbf{k}$ and S is the surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = x + 2$ and $z = 0$.
(Hint: Divergence Theorem.)

By the Divergence Thm, $I = \iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div}(\vec{F})\, dV$

$$\operatorname{div}(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}[x^4] + \frac{\partial}{\partial y}[-x^3z^2] + \frac{\partial}{\partial z}[4xy^2z]$$

$$= 4x^3 + 4xy^2$$

So $I = \iiint_E 4x^3 + 4xy^2 \, dV$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \left(\int_{z=0}^{x+2} (4x^3 + 4xy^2)\, dz \right) r\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (4x^3 + 4xy^2) \left[z \right]_0^{x+2} r\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (4x^3 + 4xy^2)(x+2)\, r\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \left(4x^4 + 8x^3 + 4x^2y^2 + 8xy^2\right) r\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \left(4r^4\cos^4\theta + 8r^3\cos^3\theta + 4r^4\cos^2\theta\sin^2\theta + 8r^3\cos\theta\sin^2\theta\right) r\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} 4r^5\cos^2\theta \overbrace{\cancel{(\cos^2\theta + \sin^2\theta)}}^{1} + 8r^4\cos\theta \overbrace{\cancel{(\cos^2\theta + \sin^2\theta)}}^{1}\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{4}{6} \overbrace{\cancel{\left[r^6\right]_0^1}}^{1} \cos^2\theta + \frac{8}{5} \overbrace{\cancel{\left[r^5\right]_0^1}}^{1} \cos\theta\, d\theta$$

$$= \left(\frac{2}{3}\right)\left(\frac{1}{2}\right) \int_0^{2\pi} 1 + \cos(2\theta)\, d\theta + \left(\frac{8}{5}\right)\left(\frac{1}{2}\right) \overbrace{\cancel{\int_0^{2\pi} \cos\theta\, d\theta}}^{0}$$

$$= \left(\frac{1}{3}\right)(2\pi) = \boxed{\frac{2\pi}{3}}$$

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Question 4: Find the general solution (you may leave your answer in implicit form):

$$2xy^3 + e^x + [3x^2y^2 + \sin(y)]y' = 0$$

$$\underbrace{(2xy^3 + e^x)}_{M} dx + \underbrace{[3x^2y^2 + \sin(y)]}_{N} dy = 0$$

$M_y = 6xy^2 = N_x$, so equation is exact.

$F = \int M dx = \int 2xy^3 + e^x dx = x^2y^3 + e^x + g(y)$

$N = F_y \Rightarrow 3x^2y^2 + g'(y) = 3x^2y^2 + \sin(y)$

$\Rightarrow g'(y) = \sin(y)$

$\Rightarrow g(y) = -\cos(y) + C_1$.

$\therefore F(x,y) = \boxed{x^2y^3 + e^x - \cos(y) + C_1 = 0}$

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Question 5: Find the general solution (state your answer in explicit form):

$$2xy^2 + x^2y' = y^2$$

$y' = \frac{y^2 - 2xy^2}{x^2} = \frac{y^2(1-2x)}{x^2}$

$\Rightarrow \frac{dy}{dx} = y^2\left(\frac{1-2x}{x^2}\right)$ } separable

$\int y^{-2} dy = \int (x^{-2} - \frac{2}{x}) dx$

$\frac{-1}{y} = \frac{-1}{x} - 2\ln|x| + C$

$\boxed{y = \left(\frac{1}{x} + 2\ln|x| + C_1\right)^{-1}}$

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Question 6: Find the general solution (state your answer in explicit form):

$$xy' + 2y = 6x^2\sqrt{y}$$

$$(*) \left\{ y' + \left(\frac{2}{x}\right) y = 6 x y^{1/2} \right\} \text{Bernoulli}, \quad n = \frac{1}{2}.$$

Let $v = y^{1-\frac{1}{2}} = y^{1/2} \Rightarrow y = v^2, \quad \frac{dy}{dx} = 2v\frac{dv}{dx}.$

$(*)$ becomes $2v v' + \left(\frac{2}{x}\right) v^2 = 6xv$

$$\Rightarrow \quad v' + \left(\frac{1}{x}\right) v = 3x$$

$$\rho = \exp\left[\int \frac{1}{x} dx\right] = \exp\left[\ln x\right] = x$$

$$\Rightarrow \quad \frac{d}{dx}\left[xv\right] = (3x)(x) = 3x^2$$

$$\Rightarrow \quad xv = x^3 + C$$

$$\Rightarrow \quad v = x^2 + \frac{C}{x}$$

$$\Rightarrow \quad y^{1/2} = x^2 + \frac{C}{x}$$

$$\therefore \boxed{y = \left(x^2 + \frac{C}{x}\right)^2}$$

[5]

Question 7: Find the general solution (state your answer in explicit form):

$$x^2y' = xy + 3y^2$$

$$\div x^2 : \quad y' = \left(\frac{y}{x}\right) + 3\left(\frac{y}{x}\right)^2 \quad \Big\} (*) \text{ Homogeneous}$$

Let $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

So $(*)$ becomes $\cancel{v} + x\frac{dv}{dx} = \cancel{v} + 3v^2$

$$\Rightarrow \quad v^{-2} dv = \frac{3}{x} dx$$

$$\Rightarrow \quad -\frac{1}{v} = 3\ln|x| + C_1$$

$$\Rightarrow \quad -\frac{x}{y} = 3\ln|x| + C$$

$$\boxed{y = \frac{-x}{3\ln|x| + C_1}}$$

[5]
