Question 1: At time $t = 1$ a particle is located at position $(1, 3)$ and moves according to the velocity field $\mathbf{F}(x, y) = \langle xy - 2, y^2 - 10\rangle$. Estimate the particle's position at time $t = 1.05$.

At $t=1$, $\vec{r}(1) = \langle 1,3\rangle$ and $\vec{r}\,'(1) = \vec{F}(1,3) = \langle (1)(3)-2, 3^2-10\rangle = \langle 1,-1\rangle$.

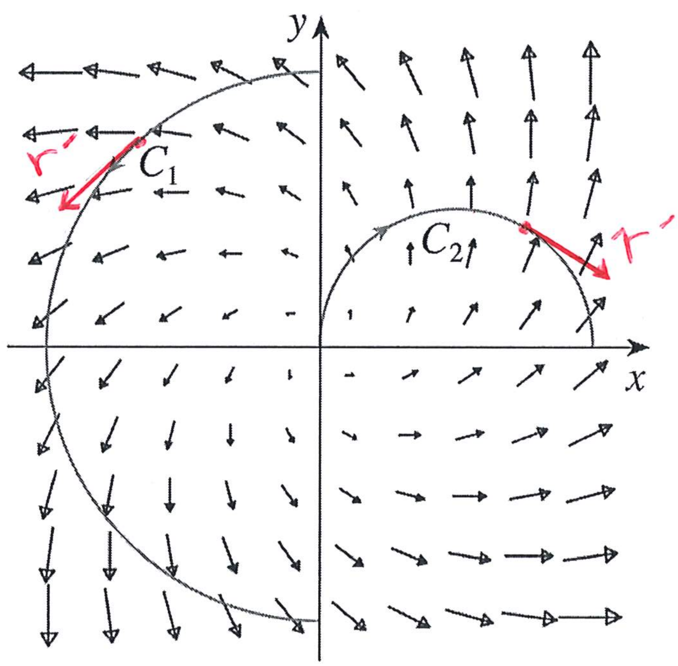
So $\vec{r}(1+\Delta t) \approx \vec{r}(1) + \vec{r}\,'(1)\Delta t$

$\vec{r}(1.05) \approx \langle 1,3\rangle + \langle 1,-1\rangle (0.05)$

$= \boxed{\langle 1.05, 2.95\rangle}$

[4]

Question 2: For this question use the following plot of the vector field $\mathbf{F}(x, y)$:



(a) Is $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ positive, negative or zero? Give a brief explanation to support your answer.

$\boxed{\text{Positive}}$: Along C_1 the angles between $\vec{F}$ and $\vec{r}\,'$ are less than $\frac{\pi}{2}$, so $\vec{F}\cdot\vec{r}\,' > 0$, so $\int_{C_1} \vec{F}\cdot d\vec{r} > 0$.

[2]

(b) Is $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ positive, negative or zero? Again, give a brief explanation to support your answer.

$\boxed{\text{Negative.}}$: $|\vec{F}|$ increases along C_2, and starting about halfway along C_2, the angles between $\vec{F}$ and $\vec{r}\,'$ exceed $\frac{\pi}{2}$, so that $\vec{F}\cdot\vec{r}\,' < 0$ along this part of the curve. Over the first half of C_2 where $\vec{F}\cdot\vec{r}\,' > 0$, $|\vec{F}|$ is small and $\vec{F}$ is nearly perpendicular to $\vec{r}\,'$, resulting in a comparatively smaller contribution to the integral.

[2]

Question 3: Determine $\int_C xyz^2\,ds$ where C is the line segment from $(-1,5,0)$ to $(1,6,4)$.

$$\vec{r}(t) = (1-t)\langle -1,5,0\rangle + t\langle 1,6,4\rangle = \langle -1+2t,\ 5+t,\ 4t\rangle,\quad 0 \le t \le 1$$

$$\therefore \int_C xyz^2\,ds = \int_0^1 (-1+2t)(5+t)(4t)^2\sqrt{2^2+1^2+4^2}\,dt$$

$$= 16\sqrt{21}\int_0^1 2t^4+9t^3-5t^2\,dt$$

$$= 16\sqrt{21}\left(\frac{2}{5}\left[t^5\right]_0^1 + \frac{9}{4}\left[t^4\right]_0^1 - \frac{5}{3}\left[t^3\right]_0^1\right)$$

$$= 16\sqrt{21}\left(\frac{2}{5}+\frac{9}{4}-\frac{5}{3}\right)$$

$$= {}^{4}\cancel{16}\sqrt{21}\left(\frac{24+135-100}{\cancel{60}_{15}}\right) = \boxed{\frac{236\sqrt{21}}{15}}$$

[5]

Question 4: Determine the work done by the force field $\mathbf{F}(x,y) = \langle x^2, ye^x\rangle$ on a particle that moves along the parabola $x = y^2+1$ from $(1,0)$ to $(2,1)$.

$$\vec{r}(t) = \langle t^2+1,\ t\rangle,\quad 0 \le t \le 1.$$

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_0^1 \langle (t^2+1)^2,\ te^{t^2+1}\rangle \cdot \langle 2t, 1\rangle\,dt$$

$$= \int_0^1 (t^2+1)^2(2t) + te^{t^2+1}\,dt$$

$$= \left[\frac{(t^2+1)^3}{3}\right]_0^1 + \left[\frac{e^{t^2+1}}{2}\right]_0^1$$

$$= \left[\frac{8}{3}-\frac{1}{3}\right] + \left[\frac{e^2}{2}-\frac{e}{2}\right] = \boxed{\frac{7}{3}+\frac{e^2-e}{2}}$$

[5]

Question 5: Let $\mathbf{F}(x,y,z) = \sin(y)\,\mathbf{i} + [x\cos(y) + \cos(z)]\,\mathbf{j} - y\sin(z)\,\mathbf{k}$.

(a) Find a potential function f for $\mathbf{F}$ (that is, a function f such that $\nabla f = \mathbf{F}$).

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle \sin(y),\ x\cos(y)+\cos(z),\ -y\sin(z) \rangle$$

$$\therefore f_x = \sin(y) \Rightarrow f = \int \sin(y)\,dx = x\sin(y) + g(y,z)$$

$$f_z = -y\sin(z) \Rightarrow \frac{\partial}{\partial z}\left[x\sin(y) + g(y,z)\right] = -y\sin(z)$$

$$\Rightarrow g_z(y,z) = -y\sin(z)$$

$$\Rightarrow g(y,z) = y\cos(z) + h(y)$$

$$\therefore f = x\sin(y) + y\cos(z) + h(y).$$

$$f_y = x\cos(y) + \cos(z) \Rightarrow \frac{\partial}{\partial y}\left[x\sin(y) + y\cos(z) + h(y)\right] = x\cos(y) + \cos(z)$$

$$\Rightarrow h'(y) = 0 \Rightarrow h(y) = K.$$

$$\boxed{\therefore f(x,y,z) = x\sin(y) + y\cos(z) + K}$$

[5]

(b) Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is given by $\mathbf{r}(t) = \langle \sin(t), t, 2t \rangle$, $0 \le t \le \pi/2$.

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (\nabla f) \cdot d\vec{r}$$

$$= f(\vec{r}(\tfrac{\pi}{2})) - f(\vec{r}(0))$$

$$= f(1, \tfrac{\pi}{2}, \pi) - f(0,0,0)$$

$$= (1)\sin(\tfrac{\pi}{2}) + (\tfrac{\pi}{2})\cos(\pi) - (0)\sin(0) - (0)\cos(0)$$

$$= \boxed{1 - \tfrac{\pi}{2}}$$

[3]

Question 6: Is $\mathbf{F}(x,y) = e^x\cos(y)\,\mathbf{i} + e^x\sin(y)\,\mathbf{j}$ conservative? Explain.

$= \langle P, Q \rangle.$

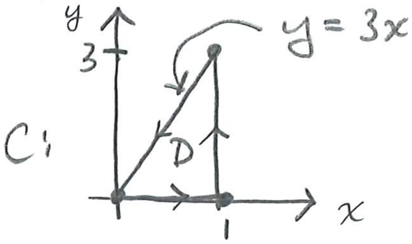
If so then $(Q_x - P_y) = 0$.

But $Q_x - P_y = e^x\sin(y) + e^x\sin(y) \neq 0$,

so $\vec{F}$ is <u>not</u> conservative.

[2]

Question 7: Evaluate $\int_C \sqrt{1+x^3}\,dx + 2xy\,dy$ where C is the positively oriented triangle with vertices $(0,0)$, $(1,0)$ and $(1,3)$.



Let $P = \sqrt{1+x^3}$, $Q = 2xy$

$$\int_C \sqrt{1+x^3}\,dx + 2xy\,dy$$

$$= \int_C P\,dx + Q\,dy$$

$$= \iint_D (Q_x - P_y)\,dA \quad \text{by Green's Thm.}$$

$$= \int_{x=0}^{1} \int_{y=0}^{3x} (2y - 0)\,dy\,dx$$

$$= \int_0^1 \left[y^2\right]_0^{3x} dx$$

$$= \int_0^1 9x^2\,dx$$

$$= \frac{9}{3}\left[x^3\right]_0^1$$

$$= \boxed{3}$$

[7]

Question 8: Let $\mathbf{F} = \langle f(x), g(y) \rangle$ where f and g are differentiable and suppose C is a simple smooth positively oriented closed curve. Determine $\int_C \mathbf{F} \cdot d\mathbf{r}$.

$$\int_C \vec{F} \cdot d\vec{r} = \int_C f(x)\,dx + g(y)\,dy$$

$$= \iint_D \Big(\cancelto{0}{\frac{\partial g(y)}{\partial x}} - \cancelto{0}{\frac{\partial f(x)}{\partial y}} \Big)\,dA \quad \text{by Green's Thm.}$$

$$= \boxed{0}$$

[3]

Question 9: Let $\mathbf{F}(x, y, z) = e^{-x}\sin(y)\,\mathbf{i} + e^{-y}\sin(z)\,\mathbf{j} + e^{-z}\sin(x)\,\mathbf{k}$. Calculate

(a) div(**F**) $= \nabla \cdot \vec{F} = \frac{\partial}{\partial x}\left[e^{-x}\sin(y)\right] + \frac{\partial}{\partial y}\left[e^{-y}\sin(z)\right] + \frac{\partial}{\partial z}\left[e^{-z}\sin(x)\right]$

$$= \boxed{-e^{-x}\sin(y) - e^{-y}\sin(z) - e^{-z}\sin(x)}$$

[2]

(b) curl(**F**) $= \nabla \times \vec{F}$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^{-x}\sin(y) & e^{-y}\sin(z) & e^{-z}\sin(x) \end{vmatrix}$$

$$= \boxed{-e^{-y}\cos(z)\,\hat{\imath} - e^{-z}\cos(x)\,\hat{\jmath} - e^{-x}\cos(y)\,\hat{k}}$$

[2]

Question 10: Is $\mathbf{F}(x, y, z) = xyz^2\,\mathbf{i} + x^2yz^2\,\mathbf{j} + x^2y^2z\,\mathbf{k}$ conservative? Explain.

$$\text{curl}(\vec{F}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ xyz^2 & x^2yz^2 & x^2y^2z \end{vmatrix}$$

$$= \langle 2x^2yz - 2x^2yz,\ 2xyz - 2xy^2z,\ 2xyz^2 - xz^2 \rangle$$

$\neq 0$, so $\boxed{\text{not conservative}}$

[3]

Question 11: Let $\mathbf{F} = \langle xyz, -y^2z, yz^2 \rangle$. Could $\mathbf{F} = \text{curl}(\mathbf{G})$ for some vector field **G**? Explain.

If so, then $\text{div}(\vec{F}) = \text{div}(\text{curl}(\vec{G})) = 0$.

But $\text{div}(\vec{F}) = \frac{\partial}{\partial x}[xyz] + \frac{\partial}{\partial y}[-y^2z] + \frac{\partial}{\partial z}[yz^2]$

$$= yz - \cancel{2yz} + \cancel{2yz}$$

$$\neq 0$$

So $\boxed{\text{no}}$, $\vec{F}$ can't be the curl of some vector field $\vec{G}$.

[3]