Question 1(a) (Integration by Parts) Determine $\int x^2 \cos(\pi x)\, dx = I$

Let $u = x^2 \quad dv = \cos(\pi x)dx$

$du = 2x\,dx \quad v = \frac{\sin(\pi x)}{\pi}$

$I = \int u\,dv$

$= uv - \int v\,du$

$= \frac{x^2\sin(\pi x)}{\pi} - \int \frac{\sin(\pi x)}{\pi} \cdot 2x\,dx$

$= \frac{x^2\sin(\pi x)}{\pi} - \frac{2}{\pi}\underbrace{\int x\sin(\pi x)\,dx}$

$u = x \quad dv = \sin(\pi x)dx$

$du = dx \quad v = -\frac{\cos(\pi x)}{\pi}$

$= \frac{x^2\sin(\pi x)}{\pi} - \frac{2}{\pi}\int u\,dv$

$= \frac{x^2\sin(\pi x)}{\pi} - \frac{2}{\pi}\left[uv - \int v\,du\right]$

$= \frac{x^2\sin(\pi x)}{\pi} - \frac{2}{\pi}\left[\frac{-x\cos(\pi x)}{\pi} - \int \frac{-\cos(\pi x)}{\pi}dx\right]$

$= \boxed{\frac{x^2\sin(\pi x)}{\pi} + \frac{2}{\pi^2}x\cos(\pi x) - \frac{2}{\pi^3}\sin(\pi x) + C}$

[5]

Question 1(b) (Integration by Parts) Determine $\int_0^1 x\tan^{-1}(x)\, dx = I$

$u = \tan^{-1}(x) \quad dv = x\,dx$

$du = \frac{1}{1+x^2} \quad v = \frac{x^2}{2}.$

$I = \int_0^1 u\,dv$

$= uv\Big|_0^1 - \int_0^1 v\,du$

$= \tan^{-1}(x)\cdot\frac{x^2}{2}\Big|_0^1 - \int_0^1 \frac{x^2}{2(1+x^2)}dx$

$= \left[\frac{x^2\tan^{-1}(x)}{2}\right]_0^1 - \frac{1}{2}\int_0^1 \frac{1+x^2-1}{1+x^2}dx$

$= \left[\frac{x^2\tan^{-1}(x)}{2}\right]_0^1 - \frac{1}{2}\int_0^1 1 - \frac{1}{1+x^2}dx$

$= \left[\frac{x^2\tan^{-1}(x)}{2}\right]_0^1 - \frac{1}{2}[x]_0^1 + \frac{1}{2}\left[\tan^{-1}(x)\right]_0^1$

$= \frac{(1)(\pi/4)}{2} - \frac{1}{2} + \frac{1}{2}\left(\frac{\pi}{4}\right)$

$= \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8}$

$= \frac{\pi}{4} - \frac{1}{2}$

$= \boxed{\frac{\pi - 2}{4}}$

[5]

Question 2: (Trigonometric Substitution) Determine $\int \frac{\sqrt{9+x^2}}{x^4}\,dx = I$

Let $x = 3\tan\theta$

$dx = 3\sec^2\theta \, d\theta$

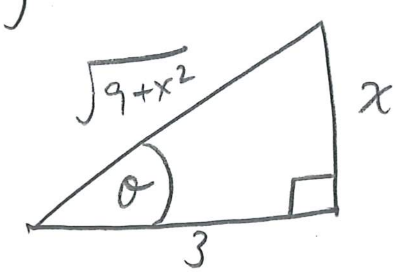
$$I = \int \frac{\sqrt{9 + 9\tan^2\theta}\;\, 3\sec^2\theta}{3^4\tan^4\theta}\,d\theta$$

$$= \int \frac{\sqrt{9(1+\tan^2\theta)}\;\, 3\sec^2\theta}{3^4\tan^4\theta}\,d\theta$$

$$= \frac{1}{9}\int \frac{\sqrt{\sec^2\theta}\;\sec^2\theta}{\tan^4\theta}\,d\theta$$

$$= \frac{1}{9}\int \frac{\cos^4\theta \cdot \sec^3\theta}{\sin^4\theta}\,d\theta$$

$$= \frac{1}{9}\int \frac{\cos\theta}{\sin^4\theta}\,d\theta \quad \left\{ \begin{array}{l} u = \sin\theta \\ du = \cos\theta\, d\theta \end{array} \right.$$



$$= \frac{1}{9}\,\frac{(\sin\theta)^{-3}}{-3} + C$$

$$= -\frac{1}{27}\,\frac{1}{\sin^3\theta} + C$$

$$= \frac{-1}{27}\,\frac{\left(\sqrt{9+x^2}\right)^3}{x^3} + C = \boxed{-\frac{1}{27}\,\frac{(9+x^2)^{3/2}}{x^3} + C}$$

[10]

Question ~~4~~ 3: (Partial Fractions) Determine $\displaystyle\int \frac{7x^2-3x+5}{x(x^2+1)}\,dx = I$

$$\frac{7x^2-3x+5}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$= \frac{(A+B)x^2 + Cx + A}{x^2+1}$$

$$\left.\begin{aligned} \therefore\ A+B &= 7 \\ C &= -3 \\ A &= 5 \end{aligned}\right\} \quad \begin{aligned} \therefore\ B &= 7-A \\ &= 7-5 \\ &= 2 \end{aligned}$$

$$\text{So}\quad I = \int \frac{5}{x} + \frac{2x-3}{x^2+1}\,dx$$

$$= \int \frac{5}{x}\,dx + \underbrace{\int \frac{2x}{x^2+1}\,dx}_{\substack{u = x^2+1 \\ du = 2x\,dx}} - 3\int \frac{1}{x^2+1}\,dx$$

$$= \boxed{5\ln|x| + \ln|x^2+1| - 3\tan^{-1}(x) + C}$$

[10]

Question 4: The velocity of an object moving along a line was measured at five points in time. The resulting data is

t (s)	0	1	2	3	4
v (m/s)	0	2	-1	0	3

Use T_4, the Trapezoid Rule on four subintervals to estimate the total change in position of the object over the four second time interval.

$$s(4) - s(0) = \int_0^4 v(t)\,dt$$

$$\approx T_4$$

$$= \frac{\Delta t}{2}\left[v(0) + 2v(1) + 2v(2) + 2v(3) + v(4)\right]$$

$$= \frac{1}{2}\left[0 + (2)(2) + (2)(-1) + (2)(0) + 3\right]$$

$$= \boxed{\frac{5}{2} \text{ m}}$$

[5]

Question 5: Determine whether the improper integral $\int_1^e \frac{1}{x\sqrt{\ln(x)}}\,dx$ converges or diverges. If it converges give the value, if it diverges then say so. Make proper use of any required limits and use proper notation.

For $I = \int \frac{1}{x\sqrt{\ln x}}\,dx$, let $u = \ln x$, $du = \frac{1}{x}\,dx$,

so $I = \int u^{-\frac{1}{2}}\,du = 2u^{1/2} + C = 2\sqrt{\ln x} + C.$

$$\therefore \int_1^e \frac{1}{x\sqrt{\ln x}}\,dx = \lim_{a\to 1^+} \int_a^e \frac{1}{x\sqrt{\ln(x)}}\,dx$$

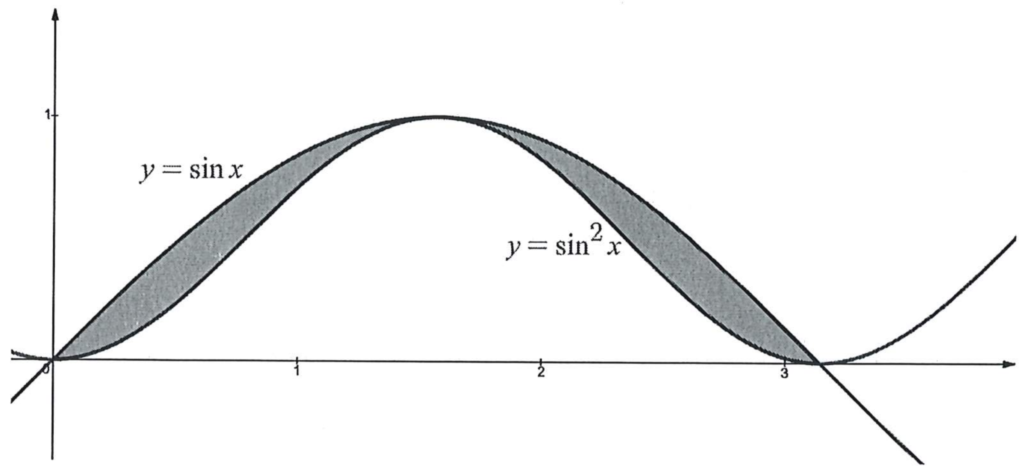
$$= \lim_{a\to 1^+} \left[2\sqrt{\ln x}\right]_a^e$$

$$= \lim_{a\to 1^+} \left[2\sqrt{\ln e} - 2\sqrt{\ln a}\right]$$

$$= \boxed{2}$$

[5]

Question 6: Determine the area of the shaded region:



$$A = \int_0^{\pi} \sin x - \sin^2 x \, dx$$

$$= \int_0^{\pi} \sin x - \frac{1-\cos(2x)}{2} \, dx$$

$$= \int_0^{\pi} \sin(x)\,dx - \int_0^{\pi} \frac{1}{2}\,dx + \frac{1}{2}\int_0^{\pi} \cos(2x)\,dx$$

$$= \big[-\cos(x)\big]_0^{\pi} - \frac{1}{2}\big[x\big]_0^{\pi} + \frac{1}{4}\big[\sin(2x)\big]_0^{\pi}$$

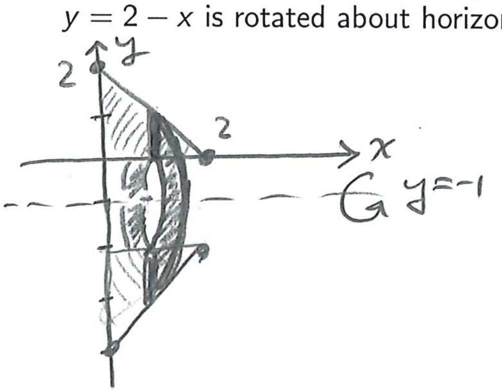
$$= -\cos(\pi) + \cos(0) - \frac{1}{2}(\pi - 0) + \frac{1}{4}\big[\underbrace{\sin(2\pi)}_{0} - \underbrace{\sin(0)}_{0}\big]$$

$$= 2 - \frac{\pi}{2} + 0$$

$$= \boxed{\frac{4-\pi}{2}}$$

[5]

Question 7: The triangular region in the first quadrant that is bounded by the x-axis, the y-axis and the line $y = 2 - x$ is rotated about horizontal line $y = -1$. Determine the volume of the resulting solid.



$$V = \int_{x=0}^{2} \pi\left[(3-x)^2 - 1^2\right] dx$$

$$= \pi\left[\frac{(3-x)^3}{-3}\right]_0^2 - \pi\big[x\big]_0^2$$

$$= \frac{\pi}{-3} + \frac{27\pi}{3} - \frac{6\pi}{3}$$

$$= \boxed{\frac{20\pi}{3}}$$

[5]