Question 1:

(a) Use the definition of the definite integral in the form

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

to evaluate

$$\int_0^3 \left(2x^3 - 3x - 4\right)\,dx$$

Carefully set up the Riemann sum and clearly show the steps of your simplification.

$$\Delta x = \frac{3-0}{n} = \frac{3}{n}$$

$$x_i = 0 + i\Delta x = \frac{3i}{n}$$

$$f(x_i) = 2(x_i)^3 - 3(x_i) - 4 = 2\left(\frac{3i}{n}\right)^3 - 3\left(\frac{3i}{n}\right) - 4 = 54\frac{i^3}{n^3} - 9\frac{i}{n} - 4$$

$$\int_0^3 (2x^3 - 3x - 4)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left(54\frac{i^3}{n^3} - 9\frac{i}{n} - 4\right)\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{162\,i^3}{n^4} - \frac{27\,i}{n^2} - \frac{12}{n}\right)$$

$$= \lim_{n\to\infty}\left[\frac{162}{n^4}\left(\sum_{i=1}^{n} i^3\right) - \frac{27}{n^2}\left(\sum_{i=1}^{n} i\right) - \frac{1}{n}\left(\sum_{i=1}^{n} 12\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{162}{n^4}\,\frac{n^2(n+1)^2}{2^2} - \frac{27}{n^2}\,\frac{n(n+1)}{2} - \frac{1}{n}\cdot 12\cdot n\right]$$

$$= \lim_{n\to\infty}\left[\frac{162}{4}\underbrace{\left(\frac{n}{n}\right)\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{n+1}{n}\right)}_{\to 1} - \frac{27}{2}\underbrace{\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)}_{\to 1} - 12\right]$$

$$= \frac{162-54-48}{4} = \boxed{15}$$

[8]

(b) Check your answer to (a) above by using the evaluation theorem to compute $\int_0^3 \left(2x^3 - 3x - 4\right)\,dx$.

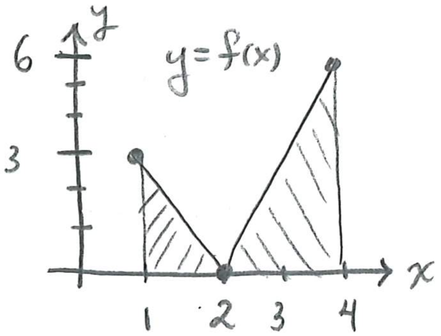
$$\int_0^3 (2x^3 - 3x - 4)\,dx = \frac{2}{4}\left[x^4\right]_0^3 - \frac{3}{2}\left[x^2\right]_0^3 - 4\left[x\right]_0^3$$

$$= \frac{81}{2} - \frac{27}{2} - 12$$

$$= \frac{81-27-24}{2} = \boxed{15}$$

[2]

Question 2: Determine the average value of $f(x) = |3x - 6|$ over the interval $[1, 4]$.



$$f_{ave} = \frac{1}{4-1}\int_1^4 f(x)\,dx$$

$$= \frac{1}{3}\left[\left(\tfrac{1}{2}\right)(1)(3) + \left(\tfrac{1}{2}\right)(2)(6)\right]$$

$$= \boxed{\frac{5}{2}}$$

[5]

Question 3: Suppose $f(x) = 2 - \int_2^{x^2+1} \frac{9}{1+t}\,dt$. Determine $f(1) - f'(1)$.

$$f(1) = 2 - \underbrace{\int_2^2 \frac{9}{1+t}\,dt}_{=0} = 2$$

$$f'(x) = 0 - \frac{9}{1+(x^2+1)} \cdot (2x) = \frac{-18x}{2+x^2}$$

chain!

$$\therefore f'(1) = \frac{-18(1)}{2+1^2} = -6$$

So $f(1) - f'(1) = 2 - (-6) = \boxed{8}$

[5]

Question 4: A tank that is initially full of water at time $t = 0$ has the water pumped out at a rate of $r(t) = 200 - 2t^2$ litres per minute until it is empty. How much water was initially in the tank? (Hint: first think about how long it takes for the flow rate to become zero.)

Tank is empty when the flow rate becomes zero:

$r(t) = 200 - 2t^2 = 0$ at $t = 10$ min.

So initial volume is equal to the total volume pumped out over $0 \le t \le 10$:

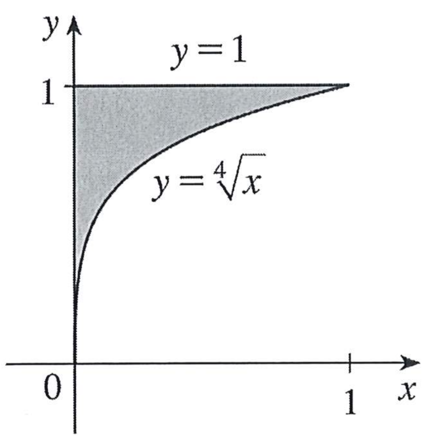
$$V = \int_0^{10} 200 - 2t^2 \, dt = 200[t]_0^{10} - \frac{2}{3}[t^3]_0^{10}$$

$$= (200)(10) - \left(\frac{2}{3}\right)(1000)$$

$$= \boxed{\frac{4000}{3} \text{ L}}$$

[5]

Question 5: Determine the area of the shaded region:



$$A = \int_0^1 1 \cdot dx - \int_0^1 x^{1/4} dx$$

$$= [x]_0^1 - \frac{4}{5}[x^{5/4}]_0^1$$

$$= 1 - \frac{4}{5}$$

$$= \boxed{\frac{1}{5}}$$

[5]

Question 6: Evaluate the following definite integrals:

(a) $\int_0^{2\pi} (1+\cos(x))\,dx$

$$= \Big[x+\sin(x)\Big]_0^{2\pi}$$

$$= \Big[2\pi + \cancelto{0}{\sin(2\pi)}\Big] - \Big[\cancelto{0}{0+\sin(0)}\Big]$$

$$= \boxed{2\pi}$$

[2]

(b) $\int_4^9 \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx$

$$= \int_4^9 x^{1/2} - x^{-1/2}\,dx$$

$$= \frac{2}{3}\Big[x^{3/2}\Big]_4^9 - 2\Big[x^{1/2}\Big]_4^9$$

$$= \frac{2}{3}\Big[9^{3/2} - 4^{3/2}\Big] - 2\Big[9^{1/2} - 4^{1/2}\Big]$$

$$= \frac{38}{3} - 2 = \boxed{\frac{32}{3}}$$

[2]

(c) $\int_0^1 (x^2+3)^2\,dx$

$$= \int_0^1 x^4 + 6x^2 + 9\,dx$$

$$= \frac{1}{5}\Big[x^5\Big]_0^1 + \frac{6}{3}\Big[x^3\Big]_0^1 + 9\Big[x\Big]_0^1$$

$$= \frac{1}{5} + 2 + 9 = \boxed{\frac{56}{5}}$$

[3]

(d) $\int_0^1 \left(\frac{4}{1+x^2}\right) dx$

$$= 4\Big[\arctan(x)\Big]_0^1$$

$$= 4\Big[\arctan(1) - \cancelto{0}{\arctan(0)}\Big]$$

$$= 4\left(\frac{\pi}{4}\right) = \boxed{\pi}$$

[3]

Question 7: (Substitution Method) Determine the following:

(a) $\frac{1}{2}\int 2xe^{x^2}\,dx = I$

Let $u = x^2$

$du = 2x\,dx$

$\therefore I = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C = \boxed{\frac{1}{2}e^{x^2} + C}$

[2]

(b) $\frac{1}{6}\int \frac{6x^2}{(2x^3+1)^4}\,dx = I$

Let $u = 2x^3+1$

$du = 6x^2\,dx$

$\therefore I = \frac{1}{6}\int u^{-4}\,du = \frac{1}{6}\frac{u^{-3}}{-3} + C = \boxed{-\frac{1}{18(2x^3+1)^3} + C}$

[2]

(c) $2\int \frac{\sin(\sqrt{x})}{2\sqrt{x}}\,dx = I$

Let $u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$\therefore I = 2\int \sin(u)\,du = -2\cos(u) + C = \boxed{-2\cos(\sqrt{x}) + C}$

[2]

(d) $\int_e^{e^2} \frac{1}{t\ln(t)}\,dt = I$

Let $u = \ln(t)$

$du = \frac{1}{t}\,dt$

$t = e \Rightarrow u = \ln(e) = 1$

$t = e^2 \Rightarrow u = \ln(e^2) = 2$

$\therefore I = \int_1^2 \frac{1}{u}\,du$

$= \left[\ln|u|\right]_1^2$

$= \ln|2| - \cancel{\ln|1|}^{\,0}$

$= \boxed{\ln(2)}$

[4]