Question 1: Evaluate the following limits, if they exist. If a limit does not exist because it is $+\infty$ or $-\infty$, state which with an explanation of your reasoning. (Do not use L'Hospital's rule to evaluate limits.)

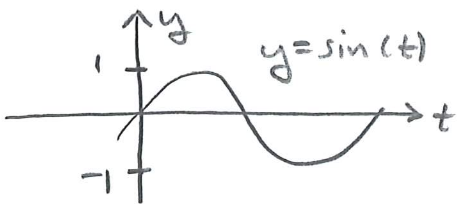
(a) $\displaystyle\lim_{x\to 2^-}\frac{x^2+x-6}{|x-2|} = \lim_{x\to 2^-}\frac{(x-2)(x+3)}{-(x-2)}$ } since $x-2<0$ as $x\to 2^-$

$\displaystyle = \lim_{x\to 2^-} -(x+3)$

$= \boxed{-5}$

[2]

(b) $\displaystyle\lim_{x\to -3^-}\frac{x+2}{\sin(x+3)}$ $\begin{matrix}\}\to -1\\ \}\to 0^-\end{matrix}$ $\displaystyle = \boxed{+\infty}$



[2]

(c) $\displaystyle\lim_{x\to -\infty}\frac{x-2x^5}{2x^2+x^5}$ $\begin{matrix}\div x^5\\ \div x^5\end{matrix}$

$\displaystyle = \lim_{x\to -\infty}\frac{(1/x^4)-2}{(2/x^3)+1}$

$= \boxed{-2}$

[3]

Question 2: Determine the value of k which makes the following function continuous at all real numbers:

$$f(x)=\begin{cases} x^2 & \text{if } x\le -1\\ k-x^3 & \text{if } x>-1\end{cases}$$

• For $x\neq -1$ $f(x)$ is defined by Polynomials, so is continuous.

• At $x=-1$ we require

$\displaystyle\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = f(-1)$

$\displaystyle\Rightarrow \lim_{x\to -1^-} x^2 = \lim_{x\to -1^+} k-x^3 = (-1)^2$

$\to$ so $(-1)^2 = k-(-1)^3 = (-1)^2$

$\Rightarrow k+1=1$

$\Rightarrow \boxed{k=0}$

[3]

Question 3:

(a) Use the limit definition of the derivative to find $f'(x)$ if $f(x) = \dfrac{x}{x+1}$. Neatly show all steps and use proper notation. (No credit will be given if $f'(x)$ is found using derivative rules.)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{x+h}{x+h+1} - \frac{x}{x+1}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{(x+h)(x+1) - x(x+h+1)}{(x+h+1)(x+1)}\right]$$

$$= \lim_{h \to 0} \frac{1}{\cancel{h}}\left[\frac{\cancel{x^2} \cancel{+hx} \cancel{+x} + \cancel{h} \cancel{-x^2} \cancel{-hx} \cancel{-x}}{(x+h+1)(x+1)}\right]$$

$$= \lim_{h \to 0} \frac{1}{(x+h+1)(x+1)} = \frac{1}{(x+1)^2}$$

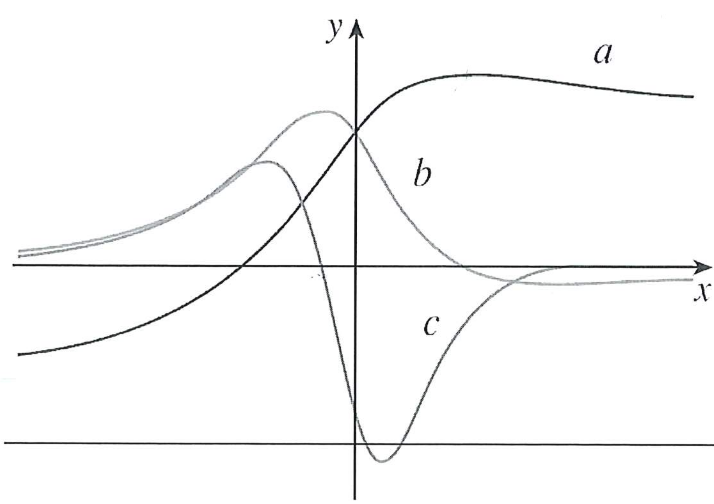
[5]

(b) At what value(s) of x will f fail to be differentiable?

f is discontinuous at $\boxed{x = -1}$ so is not differentiable there.

[2]

Question 4: The figure below shows the graphs of f, f' and f''. Identify each by circling the appropriate label.



f is graph (circle one): a b c

f' is graph (circle one): a b c

f'' is graph (circle one): a b c

[3]

Question 5: Find the following derivatives (it is not necessary to simplify your answers, but marks will be deducted for improper use of notation):

(a) $f(x) = 4\sqrt{x} - \frac{5}{x}$ $= 4x^{1/2} - \frac{5}{x}$

$f'(x) = (4)(\frac{1}{2})x^{-\frac{1}{2}} + \frac{5}{x^2}$

$= \boxed{\frac{2}{\sqrt{x}} + \frac{5}{x^2}}$

[2]

(b) $y = \sec(t)(1 - t^3)$

$y' = \sec(t)\tan(t)(1-t^3) + \sec(t)(-3t^2)$

$= \boxed{\sec(t)\tan(t)(1-t^3) - 3t^2\sec(t)}$

[3]

(c) $g(x) = \dfrac{\sin(x)}{1 + x - 3\cos(x)}$

$\boxed{g'(x) = \dfrac{[1+x-3\cos(x)]\cos(x) - \sin(x)[1+3\sin(x)]}{[1+x-3\cos(x)]^2}}$

[3]

(d) $y = \dfrac{t^2\tan(t) - 1}{\pi^2}$ $= \underbrace{\left(\frac{1}{\pi^2}\right)}_{\uparrow}[t^2\tan(t) - 1]$

multiplying constant

$\boxed{y' = \left(\frac{1}{\pi^2}\right)[2t\tan(t) + t^2\sec^2(t)]}$

[2]

Question 6: Find the following derivatives (it is not necessary to simplify your answers, but marks will be deducted for improper use of notation):

(a) $y = (1 + x^{2/3})^{3/2}$

$$y' = \frac{3}{2}\left(1+x^{2/3}\right)^{\frac{1}{2}}\left(\frac{2}{3}x^{-\frac{1}{3}}\right)$$

$$= \boxed{\frac{(1+x^{2/3})^{1/2}}{x^{1/3}}}$$

[2]

(b) $y = \dfrac{1}{2+\sqrt{3t+4}} = \dfrac{1}{2+(3t+4)^{1/2}}$

$$y' = \frac{-1}{\left[2+(3t+4)^{1/2}\right]^2} \cdot \frac{1}{2}(3t+4)^{-\frac{1}{2}} \cdot 3$$

$$= \boxed{\frac{-3}{2\left[2+(3t+4)^{1/2}\right]^2\sqrt{(3t+4)}}}$$

[2]

(c) $g(x) = \tan(x\sin(x))$

$$\boxed{g'(x) = \sec^2(x\sin(x)) \cdot [\sin(x) + x\cos(x)]}$$

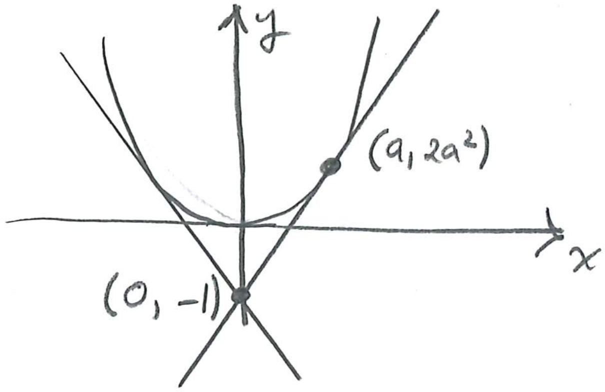
[3]

(d) $y = \sqrt[5]{\dfrac{\csc(t)}{t}} = \left(\dfrac{\csc(t)}{t}\right)^{\frac{1}{5}}$

$$\boxed{y' = \frac{1}{5}\left(\frac{\csc(t)}{t}\right)^{-\frac{4}{5}} \cdot \left[\frac{-t\csc(t)\cot(t) - \csc(t)}{t^2}\right]}$$

[3]

Question 7: There are two tangent lines to the parabola $y = 2x^2$ that pass through the point $(0, -1)$ (sketch the parabola and tangent lines to see this.) For each of these tangent lines, determine the x-coordinate of the point where the line meet the parabola.



$y' = 4x$

- Slope of tangent line is

$$m = \frac{2a^2-(-1)}{a-0} = \frac{2a^2+1}{a}$$

- But slope is also

$$\left.\frac{dy}{dx}\right|_{x=a} = \left.4x\right|_{x=a} = 4a$$

These slopes are equal, so

$$\frac{2a^2+1}{a} = 4a$$

$$2a^2+1 = 4a^2$$

$$2a^2 = 1$$

$$\therefore \boxed{a = \pm\frac{1}{\sqrt{2}}}$$

[5]

Question 8: Find an equation of the tangent line to the curve defined by $x + 2y + 1 = \dfrac{y^2}{x-1}$ at the point $(x, y) = (2, -1)$.

$$\frac{d}{dx}\left[x+2y+1\right] = \frac{d}{dx}\left[\frac{y^2}{x-1}\right]$$

$$1 + 2y' = \frac{(x-1)\,2yy' - y^2(1)}{(x-1)^2}$$

at $(2,-1)$:

$$1+2y' = \frac{(2-1)\cdot 2\cdot(-1)\,y' - (-1)^2(1)}{(2-1)^2}$$

$$\Rightarrow 1+2y' = -2y' - 1$$

$$\Rightarrow y' = \frac{-2}{4} = \frac{-1}{2}$$

So tangent line has equation

$$y - y_0 = m(x-x_0)$$

$$y-(-1) = \frac{-1}{2}(x-2)$$

$$\boxed{y+1 = \frac{-1}{2}(x-2)}$$

or

$$\boxed{y = \frac{-1}{2}x}$$

[5]