Math 121 - Basic Derivative Formulas

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Derivative Rules

Assumptions

In the following, suppose:

- c represents a constant (a fixed number)
- The functions $f(x)$ and $g(x)$ are both differentiable. That is,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \quad \text{and} \quad g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

  both exist

Constant Rule

- $\frac{d}{dx}[c] = 0$

- In words: *The derivative of a constant is zero*

- Example: $\frac{d}{dx}\left[\sqrt{2\pi}\right] = 0$

- Proof: let $f(x) = c$. Then

$$\frac{d}{dx}[c] = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

Power Rule

- If n is any real number, $\frac{d}{dx}[x^n] = nx^{n-1}$
- Example: $\frac{d}{dx}\left[x^{11}\right] = 11x^{10}$
- Proof (in the case where n is a positive integer): let $f(x) = x^n$.

$$
\begin{aligned}
\frac{d}{dx}[x^n] &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{x^n + nx^{n-1}h + (\text{terms with factor of } h^2) + \cdots - x^n}{h} \\
&= \lim_{h\to 0} nx^{n-1} + (\text{terms with factor of } h) \\
&= nx^{n-1}
\end{aligned}
$$

Power Rule, case $n = 1$

- $\frac{d}{dx}[x] = 1$
- Why? $\frac{d}{dx}\left[x^1\right] = 1 \cdot x^0 = 1$

Constant Multiple Rule

- $\dfrac{d}{dx}[c \cdot f(x)] = c \cdot \dfrac{d}{dx}[f(x)]$
- In words: *The derivative of a constant times a function is the constant times the derivative of the function*

Constant Multiple Rule

- Proof of Constant Multiple Rule:

$$\begin{aligned}
\frac{d}{dx}[c \cdot f(x)] &= \lim_{h \to 0} \frac{c \cdot f(x+h) - c \cdot f(x)}{h} \\
&= \lim_{h \to 0} c \cdot \frac{f(x+h) - f(x)}{h} \\
&= c \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= c \cdot \frac{d}{dx}[f(x)]
\end{aligned}$$

- Example: $\frac{d}{dx}\left[3x^{11}\right] = 3 \cdot \frac{d}{dx}\left[x^{11}\right] = 3(11x^{10}) = 33x^{10}$

Sum Rule

- $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$
- In words: *The derivative of a sum is the sum of the derivatives*

Sum Rule

Proof of the Sum Rule:

$$
\begin{aligned}
& \frac{d}{dx}\left[f(x)+g(x)\right] \\
= \quad & \lim_{h\to 0} \frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} \\
= \quad & \lim_{h\to 0} \frac{[f(x+h)-f(x)]+[g(x+h)-g(x)]}{h} \\
= \quad & \lim_{h\to 0} \left(\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right) \\
= \quad & \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}+\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
= \quad & \frac{d}{dx}\left[f(x)\right]+\frac{d}{dx}\left[g(x)\right]
\end{aligned}
$$

Difference Rule

- $\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$
- In words: *The derivative of a difference is the difference of the derivatives*
- Example:

$$\begin{aligned}
\frac{d}{dx}\left[x^3 - 2x^{1/2} + \frac{x}{2}\right] &= \frac{d}{dx}\left[x^3\right] - \frac{d}{dx}\left[2x^{1/2}\right] + \frac{d}{dx}\left[\frac{1}{2}x\right] \\
&= 3x^2 - 2\frac{d}{dx}\left[x^{1/2}\right] + \frac{1}{2}\frac{d}{dx}[x] \\
&= 3x^2 - 2(\frac{1}{2}x^{-1/2}) + \frac{1}{2}(1) \\
&= 3x^2 - x^{-1/2} + \frac{1}{2}
\end{aligned}$$

Sine and Cosine Rule

- $\dfrac{d}{dx}[\sin(x)] = \cos(x)$
- $\dfrac{d}{dx}[\cos(x)] = -\sin(x)$
- Example:

$$\begin{aligned} \frac{d}{dx}\left[4\cos(x) + \frac{\sin(x)}{\pi}\right] &= 4\frac{d}{dx}[\cos(x)] + \frac{1}{\pi}\frac{d}{dx}[\sin(x)] \\ &= -4\sin(x) + \frac{1}{\pi}\cos(x) \end{aligned}$$