Math 372 - Introductory Complex Variables

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Zeros and Singularities

Zeros of Holomorphic Functions

- **Definition:** A zero of a function f is a point z_0 where f is holomorphic and $f(z_0) = 0$.

- **Definition:** z_0 is a zero of order m of f if f is holomorphic at z_0 and $f(z_0) = 0$, $f'(z_0) = 0$, $f''(z_0) = 0$, $\ldots$, $f^{(m-1)}(z_0) = 0$, but $f^{(m)}(z_0) \neq 0$.

Zeros of Holomorphic Functions

- So, if f has a zero of order m at z_0, then the Taylor series for f about z_0 takes the form

$$
\begin{aligned}
f(z) &= \frac{f^{(m)}(z_0)}{m!}(z-z_0)^m + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0)^{m+1} + \cdots \\
&= (z-z_0)^m\left[a_m + a_{m+1}(z-z_0) + a_{m+2}(z-z_0)^2 + \cdots\right] \\
&= (z-z_0)^m g(z)
\end{aligned}
$$

  where $g(z)$ is holomorphic at z_0 and $g(z_0) \neq 0$ in a neighbourhood of z_0 .

- **Example:** $f(z) = \cos(z) - 1 + z^2/2$ has a zero of order 4 at $z = 0$ since

$$f(z) = \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

Isolated Singularities of Holomorphic Functions

- **Definition:** An isolated singularity of a function f is a point z_0 such that f is holomorphic in some punctured disk $0 < |z - z_0| < R$ but f is not holomorphic at z_0 itself.
- **Example:** $f(z) = \exp(z)/(z - i)$ has an isolated singularity at $z = i$.
- If f has an isolated singularity at z_0, then it has a Laurent series representation

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j$$

  in the punctured disk.
- Singularities are classified based on the form of the Laurent Series.

Isolated Singularities of Holomorphic Functions

Definition: Suppose f has an isolated singularity at z_0 and that

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j$$

on $0 < |z - z_0| < R$.

- If $a_j = 0$ for all $j < 0$, so that $f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j$, then z_0 is called a removable singularity.
- If $a_{-m} \neq 0$ for some positive integer m but $a_j = 0$ for all $j < -m$, then z_0 is called a pole of order m of f.
- If $a_j \neq 0$ for infinitely many $j < 0$ then z_0 is called an essential singularity of f.

Removable Singularities

Suppose f has a removable singularity at z_0.Then

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j$$
$$= a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

Example: $\dfrac{e^z - 1}{z} = 1 + \dfrac{z}{2!} + z^2 3! + \cdots$

- f is bounded in some punctured circular neighbourhood of z_0
- $\lim_{z \to z_0} f(z)$ exists.
- f can be redefined at $z = z_0$ so that the new function is holomorphic at z_0. Define $f(z_0) = a_0$.

Poles

Suppose f has a pole of order m at z_0. Then

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \cdots + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

Example: $\frac{\cos z}{z^2} = \frac{1}{z^2} - \frac{1}{2} + \frac{z^2}{4!} + \cdots$ has a pole of order 2 at $z = 0$

- $(z-z_0)^m f(z)$ has a removable singularity at z_0
- $\lim_{z \to z_0} |f(z)| = \infty$.
- **Lemma:** f has a pole of order m at z_0 if and only if $f(z) = g(z)/(z-z_0)^m$ in some punctured neighbourhood of z_0 where g is holomorphic and not zero at z_0.
- **Lemma:** If f has a zero of order m at z_0 then $1/f$ has a pole of order m. If f has a pole of order m at z_0, then $1/f$ has a removable singularity at z_0, and $1/f$ has a zero of order m at z_0 if we define $(1/f)(z_0) = 0$.

Essential Singularities

Suppose f has an essential singularity at z_0. Then

$$f(z) = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

Example: $\exp(1/z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$

Theorem *(Picard)*: A function with an essential singularity at z_0 assumes every complex number, with possibly one exception, as a value in any neighbourhood of z_0.

Summary

Theorem: Suppose f has an isolated singularity at z_0. Then

- z_0 is a removable singularity $\Leftrightarrow |f|$ is bounded near z_0 $\Leftrightarrow$ $\lim_{z\to z_0} f(z)$ exists $\Leftrightarrow f$ can be redefined at z_0 so that f is holomorphic at z_0 .
- z_0 is a pole $\Leftrightarrow \lim_{z\to z_0} |f(z)| = \infty \Leftrightarrow f(z) = g(z)/(z - z_0)^m$ in some punctured neighbourhood of z_0 where g is holomorphic and not zero at z_0.
- z_0 is an esential singularity $\Leftrightarrow |f(z)|$ is neither bounded near z_0 nor goes to ∞ as $z \to z_0 \Leftrightarrow f$ assumes every complex number, with possibly one exception, as a value in any neighbourhood of z_0.

Can use this theorem to classify isolated singularities without constructing the Laurent Series.