Question 1:

(i) Simplify: $\mathrm{Im}(e^{\cos(i\pi)})$

$$\cos(i\pi) = \frac{e^{i(i\pi)} + e^{-i(i\pi)}}{2} = \frac{e^{-\pi} + e^{\pi}}{2}, \text{ real}$$

$\therefore e^{\cos(i\pi)}$ is real

$$\therefore \mathrm{Im}\left(e^{\cos(i\pi)}\right) = \boxed{0}$$

[2]

(ii) Using the principal value, express in the form $a + ib$ where a and b are real: $(1+i)^{1/2}$

$$(1+i)^{\frac{1}{2}} = e^{\frac{1}{2}\mathrm{Log}(1+i)}$$
$$= e^{\frac{1}{2}[\ln(\sqrt{2}) + i\,\pi/4]}$$
$$= e^{\ln\sqrt[4]{2} + i\frac{\pi}{8}}$$
$$= \sqrt[4]{2}\cos\left(\tfrac{\pi}{8}\right) + i\sqrt[4]{2}\sin\left(\tfrac{\pi}{8}\right)$$

[3]

Question 2: Find all solutions to

(i) $e^z = i\pi$

Let $z = a+ib$.

$$e^{a+ib} = \pi e^{i\pi/2}$$

$$\Rightarrow e^a = \pi, \quad b = \frac{\pi}{2} + 2k\pi,\ k \in \mathbb{Z}$$

$$\Rightarrow z = a+ib = \boxed{\ln(\pi) + i\left[\frac{\pi}{2} + 2k\pi\right], \ k \in \mathbb{Z}}$$

[3]

(ii) $\mathrm{Log}(1+z) = \dfrac{3\pi i}{2}$

$$\Rightarrow e^{\mathrm{Log}(1+z)} = e^{i\frac{3\pi}{2}} = -i$$

$$\Rightarrow 1+z = -i$$

$$\Rightarrow \boxed{z = -i-1}$$

[2]

Question 3: Find all solutions to $\sin(z) = \frac{\sqrt{2}}{2}$.

$$\frac{e^{iz} - e^{-iz}}{2i} = \frac{\sqrt{2}}{2}$$

$$\Rightarrow e^{iz} - e^{-iz} = \sqrt{2}i$$

$$\Rightarrow e^{2iz} - \sqrt{2}i e^{iz} - 1 = 0$$

$$e^{iz} = \frac{\sqrt{2}i \pm \sqrt{-2-4(1)(-1)}}{2}$$

$$= \frac{\sqrt{2}i \pm \sqrt{2}}{2}$$

$$= (i \pm 1)\frac{\sqrt{2}}{2}$$

$$= e^{i\frac{\pi}{4}}, e^{i3\pi/4}$$

$$\boxed{\therefore z = \frac{\pi}{4} + 2k\pi, \quad \frac{3\pi}{4} + 2k\pi} \; k \in \mathbb{Z}.$$

[5]

Question 4: Let $f(z) = z^z$ be defined using the principal value of the logarithm. Compute $f'(1)$.

$$f(z) = e^{z\,\mathrm{Log}(z)}$$

$$f'(z) = e^{z\,\mathrm{Log}(z)}\left[\mathrm{Log}(z) + \overset{1}{\cancel{\frac{z}{z}}}\right]$$

$$f'(1) = e^{1 \cdot \mathrm{Log}(1)}\left[\mathrm{Log}(1) + 1\right]$$

$$= e^{[\underset{0}{\ln(1)} + i \cdot 0]}\left[\overset{0}{\ln(1)} + i \cdot 0 + 1\right]$$

$$= \boxed{1}$$

[5]

Question 5: Calculate $I = \int_\gamma \mathrm{Im}(z^2)\, dz$ where $\gamma(t) = t + \frac{i}{t}, 1 \le t \le 2$.

$$\gamma'(t) = 1 - \frac{i}{t^2}$$

$$\therefore I = \int_1^2 \mathrm{Im}\left[\left(t+\frac{i}{t}\right)^2\right]\left(1-\frac{i}{t^2}\right) dt$$

$$= \int_1^2 \mathrm{Im}\left[t^2 + 2i - \frac{1}{t^2}\right]\left(1-\frac{i}{t^2}\right) dt$$

$$= 2\int_1^2 \left(1-\frac{i}{t^2}\right) dt$$

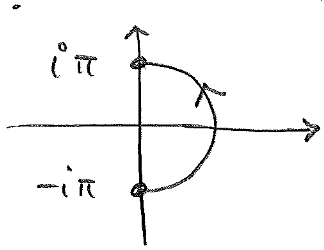
$$= 2\left[t + \frac{i}{t}\right]_1^2$$

$$= 2\left[2 + \tfrac{1}{2}i - 1 - i\right] = \boxed{2 - i}$$

[5]

Question 6: Calculate $I = \int_\gamma e^z \cos(e^z)\, dz$ where γ is the right hand side of the circle $|z| = \pi$ from $-i\pi$ to $i\pi$.

γ:



$f(z)$ is continuous on $\mathbb{C}$ with antiderivative $F(z) = \sin(e^z)$.

$$\text{Thus } I = F(i\pi) - F(-i\pi)$$

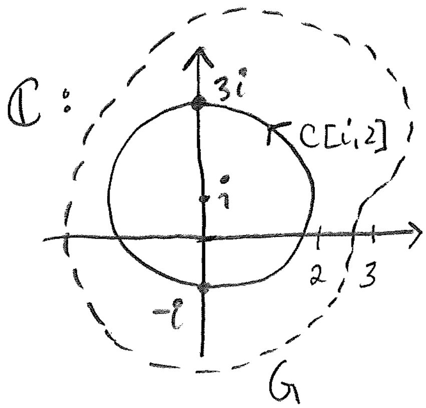
$$= \sin(e^{i\pi}) - \sin(e^{-i\pi})$$

$$= \sin(-1) - \sin(-1)$$

$$= \boxed{0}$$

[5]

Question 7: Evaluate $\displaystyle\int_{C[i,2]} \frac{\cos(z)}{z(z-3)}\,dz$ where the path $C[i,2]$ has positive orientation.



$$I = \int_{C[i,2]} \frac{f(z)}{z}\,dz$$

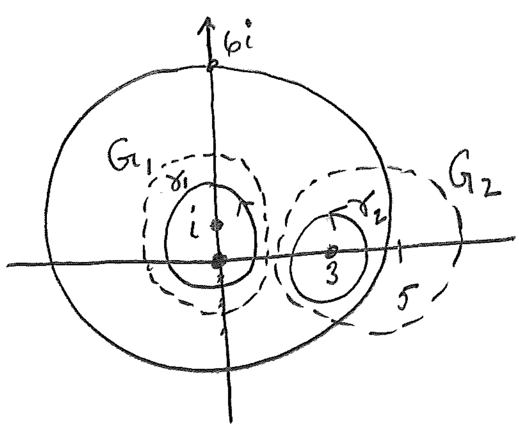
where $f(z) = \dfrac{\cos(z)}{z-3}$ is holomorphic in G containing $C[i,2]$, $C[i,2] \sim_G 0$ and 0 is inside $C[i,2]$.

By C.I.F,

$$I = 2\pi i\, f(0) = 2\pi i \frac{\cos(0)}{0-3} = \boxed{-\frac{2\pi}{3} i}$$

[5]

Question 8: Evaluate $\displaystyle I = \int_{C[i,5]} \frac{\cos(z)}{z(z-3)}\,dz$ where the path $C[i,5]$ has positive orientation.



$f(z) = \dfrac{\cos(z)}{z(z-3)}$ is holomorphic on $\mathbb{C}\setminus\{0,3\}$.

By Cauchy's Thm

$$I = \int_{\gamma_1} \frac{\cos(z)/(z-3)}{z}\,dz + \int_{\gamma_2} \frac{\cos(z)/z}{(z-3)}\,dz$$

$= I_1 + I_2$ say.

- For I_1, $f_1(z) = \dfrac{\cos(z)}{z-3}$ is holomorphic in G_1 containing γ_1, $\gamma_1 \sim_{G_1} 0$ and 0 is inside γ_1. $\therefore I_1 = 2\pi i\, f_1(0) = -\dfrac{2\pi}{3} i$.
- For I_2, $f_2(z) = \dfrac{\cos(z)}{z}$ is holomorphic in G_2 containg γ_2, $\gamma_2 \sim_{G_2} 0$ and 3 is inside γ_2. $\therefore I_2 = 2\pi i\, f_2(3) = \dfrac{2\pi}{3}\cos(3)\, i$

$\therefore I = I_1 + I_2 = \boxed{\dfrac{2\pi i}{3}\left[\cos(3) - 1\right]}$

[5]