Question 1: Simplify and express your answer in the form $a + ib$ where a and b are real:

$$\frac{2+3i}{1+2i} + \frac{8+i}{2-i}$$

$$= \frac{(2+3i)(2-i) + (8+i)(1+2i)}{(1+2i)(2-i)}$$

$$= \frac{4+4i+3+8+17i-2}{2+3i+2}$$

$$= \frac{13+21i}{4+3i} \cdot \frac{4-3i}{4-3i}$$

$$= \frac{52+45i+63}{4^2+3^2}$$

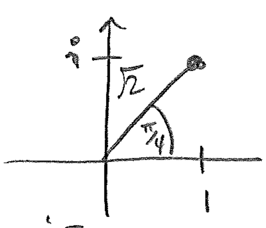
$$= \frac{115}{25} + i\,\frac{45}{25} = \boxed{\frac{23}{5} + i\,\frac{9}{5}}$$

[5]

Question 2: Express in form $z = re^{i\theta}$ where r and θ are real:

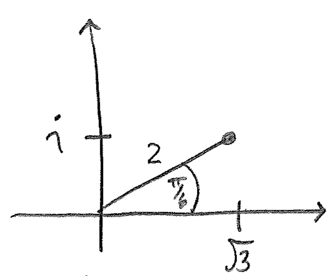
$$\frac{\sqrt{2}(1+i)}{\sqrt{3}+i}$$

$1+i$:



$1+i = \sqrt{2}\,e^{i\pi/4}$

$\sqrt{3}+i$:



$\sqrt{3}+i = 2e^{i\pi/6}$

$$\therefore \frac{\sqrt{2}(1+i)}{\sqrt{3}+i} = \frac{\cancel{\sqrt{2}} \cdot \cancel{\sqrt{2}}\, e^{i\pi/4}}{\cancel{2}\, e^{i\pi/6}} = e^{i\left(\frac{\pi}{4}-\frac{\pi}{6}\right)} = \boxed{e^{i\pi/12}}$$

[5]

Question 3: Determine all values of $\left(\dfrac{2i}{1+i}\right)^{1/2}$.

$$\frac{2i}{1+i} = \frac{2e^{i\pi/2}}{\sqrt{2}\,e^{i\pi/4}} = \sqrt{2}\,e^{i\pi/4}$$

$$\therefore \left(\frac{2i}{1+i}\right)^{\frac{1}{2}} = \left(\sqrt{2}\,e^{i\frac{\pi}{4}}\right)^{\frac{1}{2}}$$

$$= \sqrt{\sqrt{2}}\; e^{i\left(\frac{\pi}{4}+2k\pi\right)/2}, \quad k=0,1$$

$$= \boxed{\sqrt[4]{2}\; e^{i\pi/8},\ \sqrt[4]{2}\; e^{i9\pi/8}}$$

[5]

Question 4: Find all pairs (a, b) so that $f(x+iy) = ax^2 + iabxy + by^2$ is entire.

Since f is a polynomial in x & y, it will be entire provided C.R. equations are satisfied at every (x,y).

$u(x,y) = ax^2 + by^2$

$v(x,y) = abxy$

$u_x = v_y \Rightarrow 2ax = abx \Rightarrow ax(2-b) = 0$ ①

$u_y = -v_x \Rightarrow 2by = -aby \Rightarrow by(2+a) = 0$ ② .

By ①, $a=0$ or $b=2$

If $a=0$, ② $\Rightarrow b=0$

If $b=2$, ② $\Rightarrow a=-2$

$\boxed{\therefore (a,b) = (0,0) \text{ or } (-2,2)}$

[5]

Question 5:

(i) Find the points, if any, at which $f(z) = (\overline{z+i})^2$ is differentiable. For each such point explain why the function is differentiable and state the value of the derivative.

$f(z) = (\overline{z+i})^2 = (\overline{x+iy+i})^2 = (\overline{x+i(y+1)})^2 = (x-i(y+1))^2 = x^2-(y+1)^2-2ix(y+1)$

$\therefore u = x^2-(y+1)^2,\ v = -2x(y+1)$

C.R.: $\left.\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right\}\ \left.\begin{array}{l} 2x = -2x \\ -2(y+1) = +2(y+1) \end{array}\right\} \therefore x=0,\ y=-1$

$\therefore$ C.R. Equations satisfied at $z = 0+(-1)i = -i$ only and u_x, u_y, v_x, v_y continuous at $z=-i$ and exist in $D[-i,1]$, so $\boxed{f \text{ differentiable at } z=-i}$

$f'(-i) = u_x(0,-1) + iv_x(0,-1) = 0 + i\cdot 0 = \boxed{0}$.

[4]

(ii) Find the points, if any, at which $f(z) = (\overline{z+i})^2$ is holomorphic. Explain.

Since f is differentiable at the single point $z=-i$ it is not differentiable on any open disk, so is $\boxed{\text{nowhere holomorphic.}}$

[2]

Question 6: Suppose f is holomorphic in a region D and $f(z) \in \mathbb{R}$ for every $z \in D$. Explain why f must be a constant function. (Hint: consider $f = u + iv$.)

Since $f(z) = u(x,y) + iv(x,y) \in \mathbb{R}$ on D, $v(x,y) \equiv 0$ on D, so $f(z) = u(x,y)$.

Since f is holomorphic on D, C.R. equations are satisfied throughout D, so

$\left.\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right\} \Rightarrow \left.\begin{array}{l} u_x = 0 \\ u_y = 0 \end{array}\right\} \Rightarrow u(x,y) = k$, a constant.

$\therefore f(z) = k$, a constant

[4]

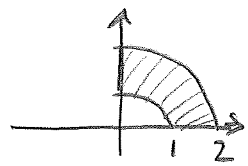
Question 7: Sketch the image of S under the mapping $f(z) = \dfrac{-1}{\overline{z}}$ where

$$S = \{z \in \mathbb{C} \mid 1 \le |z| \le 2 \text{ and } 0 \le \arg(z) \le \tfrac{\pi}{2}\}$$

For $z \in S$, $z = re^{i\theta}$ with $1 \le r \le 2$ and $0 \le \theta \le \frac{\pi}{2}$.

S:

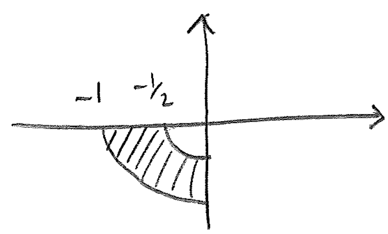
$\mathbb{C}$:



$$f(z) = \frac{-1}{\overline{z}} = \frac{-1}{re^{-i\theta}} = \frac{1}{r}(-1)e^{i\theta} = \frac{1}{r}e^{i\pi}e^{i\theta} = \underbrace{\frac{1}{r}e^{i(\theta+\pi)}}_{\frac{1}{2} \le \frac{1}{r} \le 1,\ \pi \le \theta + \pi \le \frac{3\pi}{2}}$$

$f(S)$:

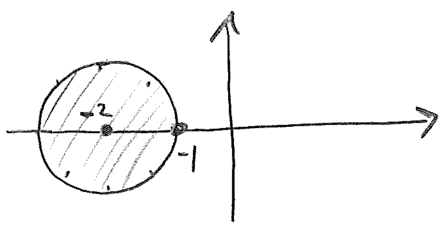
$\mathbb{C}$:



[5]

Question 8: Determine the absolute maximum value of $\left|\dfrac{1+z^2}{z^2}\right|$ on the closed disk $S = \{z \in \mathbb{C} \,:\, |z+2| \le 1\}$.

S:



$$\left|\frac{1+z^2}{z^2}\right| = \left|\frac{1}{z^2} + 1\right| \le \left|\frac{1}{z^2}\right| + |1|$$

$$= \frac{1}{|z|^2} + 1$$

$$\le \frac{1}{1^2} + 1 \quad \text{when } z = -1$$

$$= \boxed{2}$$

[5]