The final exam will be comprehensive, covering all material since the beginning of the course. In addition to reviewing your assignments and class notes, I suggest you work through the list of sample exam questions below.

Cheat Sheet and Calculator

As with tests, a single double-sided letter-size handwritten "cheat sheet" containing formulae, theory and numerical values may be used for the exam. The cheat sheet may not contain worked examples however, and must be submitted when you hand in your test.

A standard non-graphing scientific calculator may be used.

Sample Questions

1. Simplify and express in the form $a + bi$: $\dfrac{5+5i}{(1+3i)(\frac{1}{2}-\frac{i}{2})}$.
2. Determine and sketch all cube roots of $8(1-\sqrt{3}i)$.
3. Evaluate $|e^{iz}|$ if $z = 6e^{i\pi/3}$.
4. Determine the points, if any, at which $f(z) = |\bar{z} - i|^2$ is holomorphic.
5. Determine the harmonic conjugate of $u(x, y) = 3x^2y - y^3 + x + 4xy$.
6. Can $u(x, y) = xy^2$ be the real part of an entire function? Explain.
7. Find all values of $[\mathrm{Log}(i)]^{\mathrm{Log}(i)}$.
8. Find all solutions to $e^{iz} = i$.
9. Evaluate $\displaystyle\int_\gamma \mathrm{Re}(z)\,dz$ where
   (a) γ is a line from $z = 0$ to $z = 1 + i$
   (b) γ is a line segment from from $z = 0$ to $z = i$ followed by a line segment from $z = i$ to $z = 1 + i$.
10. Evaluate $\displaystyle\int_C \frac{\cos z}{e^z - 1}\,dz$ where C is the circle $|z - 2i| = 1$ traversed once in the positive direction.
11. Evaluate $\displaystyle\int_\gamma \frac{1}{z}\,dz$ where γ is any simple path from $z = -2$ to $z = -i$ which does not leave the third quadrant.
12. Evaluate $\displaystyle\int_C \frac{z^3}{(z+i)(z+2)^2}\,dz$ where C is the circle

(a) $|z| = 1/2$
(b) $|z| = 3/2$
(c) $|z + 2| = 1/2$
(d) $|z| = 3$

In each case the circle is traversed once in the positive direction.

13. Show that $f(z) = \dfrac{1 + \cos(\pi z)}{(z^2 - 1)^2}$ has a removable singularity at $z = -1$ (You may use L'Hospital's rule here.)

14. Classify the isolated singularities of the following as either removable, a pole, or essential:

(a) $\dfrac{1}{e^z - 1}$
(b) $\cos(1 - \dfrac{1}{z})$

15. Use the residue theorem to evaluate the following integrals. In each case the circles are traversed once in the positive direction:

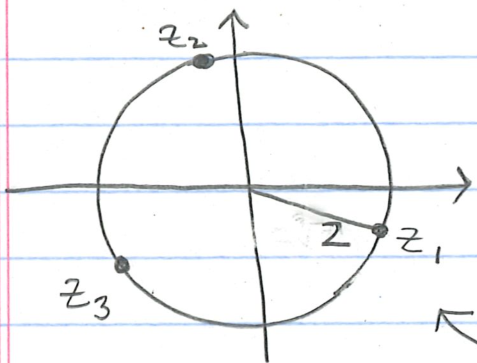
(a) $\displaystyle\int_{|z|=2} \frac{z^3 + 2z}{z - i}\, dz$
(b) $\displaystyle\int_{|z|=1} z^2 e^{1/z}\, dz$

16. Determine $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2 + 4x + 13)^2}$ using the Residue theorem.

(1) $\dfrac{5+5i}{(1+3i)(\frac{1}{2}-\frac{i}{2})} = \dfrac{10+10i}{(1+3i)(1-i)} = \dfrac{10(1+i)}{(1+3i-i+3)} = \dfrac{10(1+i)}{4+2i}\cdot\dfrac{4-2i}{4-2i} = \dfrac{10(6+2i)}{20}$
$= \boxed{3+i}$

---

(2) $z=8(1-\sqrt{3}i) = 2^3 e^{-i\pi/3} \Rightarrow z^{1/3} = \sqrt[3]{2^3}\, e^{i[-\pi/3+2k\pi]/3}, \quad k=0,1,2$



$= 2e^{i[-\frac{\pi}{9}+\frac{2k\pi}{3}]}, \quad k=0,1,2$

$= \boxed{\underbrace{2e^{-i\frac{\pi}{9}},\ 2e^{i\frac{5\pi}{9}},\ 2e^{i\frac{11\pi}{9}}}}$

$= z_1, z_2, z_3$ say.

---

(3) $z = 6e^{i\pi/3} = 6\cos(\frac{\pi}{3}) + i\,6\sin(\frac{\pi}{3}) = 6(\frac{1}{2}) + i\,6(\frac{\sqrt{3}}{2}) = 3+i\,3\sqrt{3}$.

$\therefore \left|e^{iz}\right| = \left|e^{i(3+i3\sqrt{3})}\right| = \left|e^{-3\sqrt{3}}\right|\cdot\cancel{\left|e^{i3}\right|} = \boxed{e^{-3\sqrt{3}}}$

---

(4) $f(z) = |\bar{z}-i|^2 = |\overline{x+iy}-i|^2 = |x-iy-i|^2 = x^2+(1+y)^2$

$\therefore u(x,y) = x^2+(1+y)^2, \quad v(x,y)=0$.

$\left.\begin{array}{l} u_x = 2x, \quad v_y = 0 \\ u_y = 2(1+y), \quad -v_x = 0 \end{array}\right\}$ So $\left.\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right\} \Rightarrow x=0, y=-1$.

C.R. equations are satisfied at $z=x+iy=-i$ only but not on any open disk centered at $z=-i$, so $f(z)$ is $\boxed{\text{nowhere holomorphic}}$.

$\longrightarrow$

(5) $u(x,y) = 3x^2y - y^3 + x + 4xy$

$u_x = 6xy + 1 + 4y$, $u_y = 3x^2 - 3y^2 + 4x$

$u_x = v_y \Rightarrow v = \int 6xy + 1 + 4y \, dy = 3xy^2 + y + 2y^2 + g(x)$

$u_y = -v_x \Rightarrow -\frac{\partial}{\partial x}\left[3xy^2 + y + 2y^2 + g(x)\right] = 3x^2 - 3y^2 + 4x$

$\Rightarrow -3y^2 - g'(x) = -3y^2 + 3x^2 + 4x$

$\Rightarrow g'(x) = -3x^2 - 4x$

$\Rightarrow g(x) = -x^3 - 2x^2 + C$

$\boxed{\therefore v(x,y) = 3xy^2 + y + 2y^2 - x^3 - 2x^2 + C}$

---

(6) $\boxed{\text{No.}}$ If $u(x,y) = xy^2 = \mathrm{Re}[f(z)]$ where f is entire then $u_{xx} + u_{yy} = 0$ at every $x+iy \in \mathbb{C}$.

But $u_{xx} = 0$, $u_{yy} = 2x$,

so $u_{xx} + u_{yy} = 2x \not\equiv 0$ on $\mathbb{C}$.

$\therefore$ $u(x,y) = xy^2$ cannot be the real part of an entire function.

---

(7) $\mathrm{Log}(i) = \mathrm{Log}(1 \cdot e^{i\pi/2}) = \cancel{\ln(1)}^{0} + i\left(\frac{\pi}{2}\right) = i\pi/2$.

$$
\begin{aligned}
[\mathrm{Log}(i)]^{\mathrm{Log}(i)} &= e^{\mathrm{Log}(i) \cdot \mathrm{Log}[\mathrm{Log}(i)]} \\
&= e^{(i\pi/2) \cdot \mathrm{Log}[i\pi/2]} \\
&= e^{(i\pi/2) \cdot \mathrm{Log}\left[\frac{\pi}{2} e^{i\pi/2}\right]} \\
&= e^{(i\pi/2)\left[\ln\left(\frac{\pi}{2}\right) + i\pi/2\right]} \\
&= e^{-\frac{\pi^2}{4}} e^{i\frac{\pi}{2}\ln\left(\frac{\pi}{2}\right)} \\
&= \boxed{e^{-\frac{\pi^2}{4} + i\frac{\pi}{2}\ln\left(\frac{\pi}{2}\right)}}
\end{aligned}
$$

(8) $e^{iz} = i \Rightarrow e^{i(a+ib)} = e^{i\pi/2}$

$\Rightarrow e^{-b} e^{ia} = e^{i\pi/2}$

$\Rightarrow |e^{-b}| \cancel{|e^{ia}|}^{1} = \cancel{|e^{i\pi/2}|}^{1}$

$\Rightarrow e^{-b} = 1$

$\Rightarrow b = 0$

$\therefore e^{ia} = e^{i\pi/2}$

$\Rightarrow a = \frac{\pi}{2} + 2k\pi, \ k \in \mathbb{Z}$

$\therefore \boxed{z = \frac{\pi}{2} + 2k\pi}$

<u>or:</u> $e^{iz} = i \Rightarrow iz$ is a logarithm of i

$\Rightarrow iz = \log(i)$

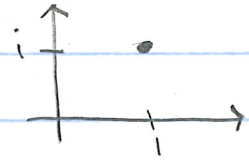
$= \log(1 \cdot e^{i\pi/2})$

$= \cancel{\ln(1)}^{0} + \frac{\pi}{2} + 2k\pi, k \in \mathbb{Z}$

$= \boxed{\frac{\pi}{2} + 2k\pi, k \in \mathbb{Z}}$

---

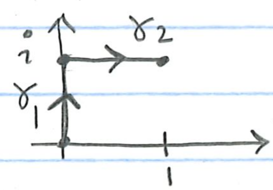
(9)(a) $I = \int_\gamma \mathrm{Re}(z)\, dz$, γ:



$\begin{cases} \gamma(t) = t + it, & 0 \le t \le 1 \\ \gamma'(t) = 1 + i \end{cases}$

$\therefore I = \int_0^1 \mathrm{Re}(t+it)(1+i)\, dt = (1+i) \int_0^1 t\, dt = (1+i)\left[\frac{t^2}{2}\right]_0^1 = \boxed{\frac{1+i}{2}}$

(9)(b) $I = \int_\gamma \mathrm{Re}(z)\, dz$, γ:



$\begin{cases} \gamma_1(t) = it, & 0 \le t \le 1 \\ \gamma_1'(t) = i \\ \gamma_2(t) = i + t, & 0 \le t \le 1 \\ \gamma_2'(t) = 1 \end{cases}$

$\therefore I = \int_{\gamma_1} + \int_{\gamma_2} = \int_0^1 \cancel{\mathrm{Re}(it)}^{0} \cdot i\, dt + \int_0^1 \mathrm{Re}(i+t) \cdot 1\, dt$

$= \int_0^1 t\, dt$

$= \frac{1}{2}[t^2]_0^1 = \boxed{\frac{1}{2}}$

(10) $I = \int_{C[2i,1]} \frac{\cos(z)}{e^z - 1}\, dz$.

$e^z - 1 = 0$ at $z = e^{i2k\pi}$, $k \in \mathbb{Z}$, so $f(z) = \frac{\cos(z)}{e^z - 1}$ is holomorphic in a region G containing $\gamma = C[2i, 1]$, and since $\gamma \sim_G 0$ it follows from Cauchy's Thm that $\boxed{I = 0}$.

(11) $I = \int_\gamma \frac{1}{z}\, dz$ where γ:

$\frac{1}{z}$ has antiderivative $\mathcal{L}_0(z)$ in the region G shown, so

$$I = \mathcal{L}_0(i) - \mathcal{L}_0(-2)$$
$$= \left[\ln|i| + i \arg_0(i)\right] - \left[\ln|-2| + i \arg_0(-2)\right]$$
$$= \left[0 + i \cdot \tfrac{3\pi}{2}\right] - \left[\ln(2) + i\pi\right]$$
$$= \boxed{-\ln(2) + i\pi/2}$$

(12)(a) $I = \int_{C[0,1/2]} \frac{z^3}{(z+i)(z+2)^2}\, dz$

$f(z) = \frac{z^3}{(z+i)(z+2)^2}$ is holomorphic on $G = C[0, 3/4]$ containing $\gamma = C[0, 1/2]$, and $\gamma \sim_G 0$, so by Cauchy's Thm, $\boxed{I = 0}$.

(12)(b) $I = \int_{C[0,\frac{3}{2}]} \frac{z^3}{(z+i)(z+2)^2}\,dz = \int_{C[0,3/2]} \frac{z^3/(z+2)^2}{z-(-i)}\,dz.$

$f(z) = \frac{z^3}{(z+2)^2}$ is holomorphic on $G = C[0,\frac{7}{4}]$ containing $\gamma = C[0,3/2]$ and $\gamma \sim_G 0$ with $-i$ inside γ.

By Cauchy's Integral Formula,

$$I = 2\pi i\, f(-i) = 2\pi i \frac{(-i)^3}{(i+2)^2} = \frac{-2\pi}{3-4i} = \boxed{\frac{-2\pi(3+4i)}{25}}$$

---

(12)(c)

$I = \int_{C[-2,\frac{1}{2}]} \frac{z^3}{(z+i)(z+2)^2}\,dz = \int_{C[-2,\frac{1}{2}]} \frac{z^3/(z+i)}{(z-(-2))^2}\,dz.$

$f(z) = \frac{z^3}{z+i}$ is holomorphic on $G = C[-2,1]$ containing $\gamma = C[-2,\frac{1}{2}]$. and $\gamma \sim_G 0$ with -2 inside γ.

By the generalized Cauchy Integral Formula,

$$I = 2\pi i\, f'(-2)$$

$$= 2\pi i \left[\frac{(z+i)3z^2 - z^3}{(z+i)^2}\right]_{z=-2}$$

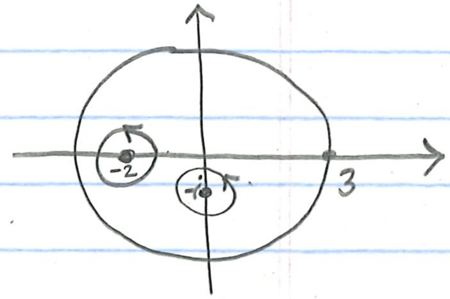
$$= 2\pi i \left[\frac{(-2+i)3\cdot(-2)^2 - (-2)^3}{(-2+i)^2}\right]$$

$$= 2\pi i \left[\frac{-16+12i}{3-4i}\right]$$

$$= 8\pi i \left(\frac{-4+3i}{3-4i} \cdot \frac{3+4i}{3+4i}\right)$$

$$= 8\pi i \left(\frac{-24-7i}{25}\right) = \boxed{\frac{56}{25}\pi - \frac{192\pi i}{25}}$$

(12)(d) $\quad I = \int_{C[0,3]} \frac{z^3}{(z+i)(z+2)^2}\, dz$



$$= \int_{C[-i,\frac{1}{2}]} + \int_{C[-2,\frac{1}{2}]}$$

$$= \int_{C[-i,\frac{1}{2}]} \frac{z^3/(z+2)^2}{z-(-i)}\, dz + \int_{C[-2,\frac{1}{2}]} \frac{z^3/(z+i)}{(z-(-2))^2}\, dz$$

$$= I_1 + I_2.$$

$$= \frac{-2\pi(3+4i)}{25} + \frac{56\pi}{25} - \frac{192\pi i}{25} \quad \text{using (12)(b)(c)}$$

$$= \boxed{2\pi - 8\pi i}$$

(13) $f(z) = \dfrac{1+\cos(\pi z)}{(z^2-1)^2}$ has a removable singularity

at $z=-1$ if $\lim\limits_{z\to -1} f(z)$ exists. (See slide #7 from "Zeros & Singularities".)

$$\lim_{z\to -1} \frac{1+\cos(\pi z)}{(z^2-1)^2} \sim \text{"}\frac{0}{0}\text{"}$$

$$\overset{H}{=} \lim_{z\to -1} \frac{-\pi \sin(\pi z)}{2(z^2-1)(2z)} \sim \text{"}\frac{0}{0}\text{"}$$

$$\overset{H}{=} \lim_{z\to -1} \frac{-\pi^2 \cos(\pi z)}{12z^2-4}$$

$$= \frac{\pi^2}{8}, \text{ so } z=-1 \text{ is } \boxed{\text{removable}}$$

---

(14)(a) $f(z) = \dfrac{1}{e^z-1}$ has isolated singularities where $e^z=1$, i.e. at $z = 2k\pi$, $k \in \mathbb{Z}$.

Since $\lim\limits_{z\to 2k\pi} |f(z)| = \infty$, these singularities are poles.

Since $\lim\limits_{z\to 2k\pi} (z-2k\pi)^1 f(z)$

$$= \lim_{z\to 2k\pi} \frac{(z-2k\pi)^1}{e^z-1} \sim \text{"}\frac{0}{0}\text{"}$$

* see slide #8 from "Zeros & Singularities"

$$= \lim_{z\to 2k\pi} \frac{1}{e^z}$$

m

1

$= 1$, $(z-2k\pi) f(z)$ has removable singularities at $z = 2k\pi$, so these are $\boxed{\text{poles of order } m=1}$

(14) (b) $f(z) = \cos\left(1-\frac{1}{z}\right)$ has an isolated singularity at $z=0$.

- Since $\lim_{z\to 0} \cos\left(1-\frac{1}{z}\right)$ does not exist, singularity is <u>not</u> removable.
- Since $\lim_{z\to 0} \left|\cos\left(1-\frac{1}{z}\right)\right| \neq \infty$, singularity is <u>not</u> a pole.

$\therefore$ Singularity is essential

(See slide #10 from "Zeros & Singularities".)

---

(15) (a) $I = \int_{\gamma = C[0,2]} \frac{z^3+2z}{z-i}\,dz = \int_\gamma \frac{z(z^2+2)}{z-i}\,dz.$

$f(z) = \frac{z(z^2+2)}{z-i}$ has an isolated singularity at $z=i$ inside γ.

Since $f(z) = \frac{g(z)}{(z-i)^1}$ where $g(z) = z(z^2+2)$ is holomorphic and not zero at $z=i$, $z=i$ is a simple pole. (see slide #10 of "Zeros & Singularities".)

$\text{Res}(f;i) = \lim_{z\to i} (z-i)f(z) = i(i^2+2) = i$

(See slide #10 of "The Residue Theorem")

$$\left.\begin{aligned} \therefore\ I &= 2\pi i\,\text{Res}(f;i) \\ &= 2\pi i \cdot (i) \\ &= \boxed{-2\pi} \end{aligned}\right\} \text{ Slide \#13 of "The Residue Thm"}$$

(15) (b) $I = \int_{\gamma = C[0,1]} z^2 e^{\frac{1}{z}} \, dz$

$f(z) = z^2 e^{\frac{1}{z}}$ has an isolated singularity at $z_0 = 0$ inside $\gamma = C[0,1]$.

To determine the residue in this case, it is easiest to write out the Laurent series for f about $z_0 = 0$:

$$e^{w} = 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots, \quad w \in \mathbb{C}.$$

Letting $w = \frac{1}{z}$;

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots, \quad z \in \mathbb{C}\setminus\{0\}$$

So

$$z^2 e^{\frac{1}{z}} = z^2 + z + \frac{1}{2} \boxed{+ \frac{1}{3!}} \frac{1}{z} + \frac{1}{4!}\frac{1}{z^2} + \cdots, \quad z \in \mathbb{C}\setminus\{0\}$$

$\operatorname{Res}(f; 0) =$ coefficient of $\frac{1}{z - z_0}$ in Laurent series expansion.

$\therefore$ by Cauchy's Residue thm (slide #13 of "The Residue Theorem"),

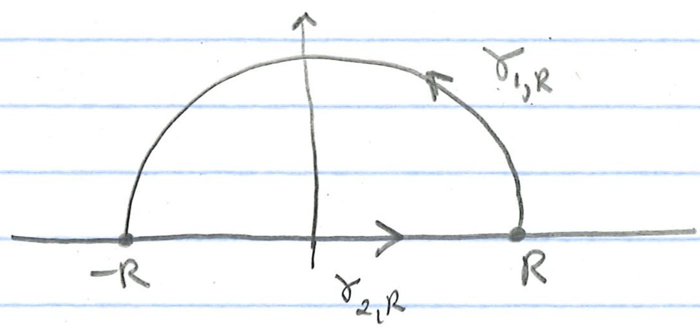
$$\begin{aligned} I &= 2\pi i \operatorname{Res}(f; 0) \\ &= 2\pi i \left(\frac{1}{3!}\right) \\ &= \boxed{\frac{\pi i}{3}} \end{aligned}$$

(16) $I = \int_{-\infty}^{\infty} \frac{x}{(x^2+4x+13)^2}\, dx$.

This is similar to the first question on the last assignment and the same approach can be used, but we'll use the residue theorem instead of Cauchy's Integral Thm.

Consider $I_R = \int_{\gamma_R} \frac{z}{(z^2+4z+13)^2}\, dz$ where

γ_R is



So $\int_{\gamma_R} = \int_{\gamma_{1,R}} + \int_{\gamma_{2,R}}$

and $I = \lim_{R\to\infty} \int_{\gamma_{2,R}}$.

We'll show that $\lim_{R\to\infty} \int_{\gamma_{1,R}} = 0$, so that

$I = \lim_{R\to\infty} \int_{\gamma_R}$.

First, calculate $\int_{\gamma_R}$ using the residue thm.

$$f(z) = \frac{z}{(z^2+4z+13)^2}$$

$$= \frac{z}{(z-(-2+3i))^2 (z-(-2-3i))^2},$$

So f has isolated singularities at $z = -2 \pm 3i$, but only $z = -2+3i$ is inside γ_R.

Furthermore, writing

$$f(z) = \frac{\overbrace{z/(z-(-2-3i))^2}^{\text{holomorphic and } \neq 0 \text{ at } z=-2+3i}}{(z-(-2+3i))^{2} \leftarrow m=2}$$

Shows that $z = -2+3i$ is a pole of order $m=2$.
(slide #10 from "zeros & Singularities")

So the residue of f at $z = -2+3i$ is

$$\text{Res}(f; -2+3i) = \lim_{z \to -2+3i} \frac{d}{dz}\left[(z-(-2+3i))^2 f(z)\right]$$

(see slide #10 from "The Residue Thm".)

$$= \lim_{z \to -2+3i} \frac{d}{dz}\left[\frac{z}{(z-(-2-3i))^2}\right]$$

$$= \lim_{z \to -2+3i} \frac{(z-(-2-3i))^{\cancel{2}} - 2z\cancel{(z-(-2-3i))}}{(z-(-2-3i))^{\cancel{4}\,3}}$$

$$= \lim_{z \to -2+3i} \frac{(z+2+3i) - 2z}{(z+2+3i)^3}$$

$$= \frac{\cancel{-2+3i} \cancel{+2+3i} + 4 - 6i}{(\cancel{-2}+3i+\cancel{2}+3i)^3}$$

$$= \frac{4i}{6^3}.$$

$\longrightarrow$

So $$I_R = \int_{\gamma_R} \frac{z}{(z^2+4z+13)^2}\, dz$$

$$= 2\pi i \operatorname{Res}\left(f; -2+3i\right)$$

$$= 2\pi i \cdot \frac{4i}{6^3}$$

$$= -\frac{8\pi}{6^3}$$

$$= \frac{-\pi}{27}$$ provided R is large enough so that $-2+3i$ is inside γ_R.

To finish off, we show that $\lim_{R\to\infty} \int_{\gamma_{1,R}} = 0$:

$$\left| \int_{\gamma_{1,R}} \frac{z}{(z^2+4z+13)^2}\, dz \right| \leq \max_{z\in\gamma_{1,R}} \left| \frac{z}{(z^2+4z+13)^2} \right| \cdot \text{length}(\gamma_{1,R})$$

$$= \max_{z\in\gamma_{1,R}} \frac{|z|}{|z-(-2+3i)|^2\,|z-(-2-3i)|^2} \cdot \pi R$$

$$\leq \max_{z\in\gamma_{1,R}} \frac{|z|}{\big||z|-|-2+3i|\big|^2 \cdot \big||z|-|-2-3i|\big|^2} \cdot \pi R$$

$$\leq \frac{R \cdot \pi R}{(R-|-2+3i|)^2(R-|-2-3i|)^2}$$

$$\to 0 \text{ as } R \to \infty.$$

Thus $$I = \lim_{R\to\infty} \int_{\gamma_R} = \lim_{R\to\infty} \frac{-\pi}{27} = \boxed{\frac{-\pi}{27}}$$