Math 372 - Introductory Complex Variables

Overview of Series

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Recap of Last Day

- **Theorem:** If f is holomorphic inside a region G containing the simple closed piecewise smooth positively oriented G-contractible path γ and z is inside γ then

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta$$

- Consequently: If f is holomorphic inside the region G, then $f'(z)$, $f''(z)$, $f'''(z)$, … all exist for $z \in G$.

Liouville's Theorem

Theorem: A bounded entire function is a constant.

Proof: Suppose that f is entire and that $|f(z)| \leq M$ for every $z \in \mathbb{C}$. Let $z \in \mathbb{C}$. Then for any $R > 0$,

$$f'(z) = \frac{1}{2\pi i} \int_{C[z,R]} \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta$$

On $C[z, R]$, $\left| \frac{f(\zeta)}{(\zeta - z)^2} \right| \leq \frac{M}{R^2}$, and the length of $C[z, R]$ is $2\pi R$.Therefore

$$|f'(z)| \leq \frac{1}{2\pi} \left(\frac{M}{R^2} \right) (2\pi R) = \frac{M}{R}$$

This is true for every $R > 0$; now let $R \to \infty$ to find $f'(z) = 0$ for every $z \in \mathbb{C}$.

Overview of Series

Series: Basic Idea

- We know from real variable theory that many functions can be expressed as infinite series.
- For example,

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \end{aligned}$$

- To what extent does this theory extend to complex variables? Many of the definitions and theorems are similar…

Definitions

- **Definition:** A series is a sum

$$c_0 + c_1 + c_2 + \cdots = \sum_{k=0}^{\infty} c_k \text{ where the terms } c_k \in \mathbb{C}$$

- The n^{th} partial sum is $S_n = \sum_{k=0}^{n} c_k$.

- If $\lim_{n\to\infty} S_n$ exists and equals S (say), we say that $\sum_{k=0}^{n} c_k$ converges to S and we write $S = \sum_{k=0}^{\infty} c_k$

- If $\lim_{n\to\infty} S_n$ does not exist say the series diverges.

The Geometric Series

Theorem: Suppose $|z| < 1$. Then $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$

Proof:

$$\begin{aligned} S_n &= 1 + z + z^2 + \cdots + z^{n-1} + z^n \\ zS_n &= \phantom{1 + {}} z + z^2 + \cdots + z^{n-1} + z^n + z^{n+1} \end{aligned}$$

$$\begin{aligned} S_n - zS_n &= 1 - z^{n+1} \\ S_n(1-z) &= 1 - z^{n+1} \\ S_n &= \frac{1 - z^{n+1}}{1-z} \end{aligned}$$

continued...

An Important Series, continued

Now let $n \to \infty$, so that $|z^{n+1}| = |z|^{n+1} \to 0$ since $|z| < 1$, leaving

$$\lim_{n\to\infty} S_n = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$$

Convergence Tests

- Many convergence results for series of real terms extend to those with complex terms and the proofs are similar.

- **The Comparison Test:** Suppose $|c_k| \leq M_k$ for all $k \geq K$ (that is, eventually all of the c_k terms have modulus bounded by real numbers M_k.)

  Then if $\sum_{k=0}^{\infty} M_k$ converges so does $\sum_{k=0}^{\infty} c_k$.

continued...

Convergence Tests, continued

- **The Ratio Test:** Suppose the series $\sum_{k=0}^{\infty} c_k$ is such that $\lim_{k\to\infty} \left| \frac{c_{k+1}}{c_k} \right| = L$. Then

  (i) If $L < 1$ the series converges

  (ii) If $L > 1$ the series diverges

  (iii) If $L = 1$ the test is inconclusive

Convergence Examples

Example: Determine the sum $\sum_{k=0}^{\infty} \frac{3}{(1+i)^k}$

Convergence Examples

Example: Determine whether $\sum_{k=0}^{\infty} \frac{k^2}{3^k}$ converges or diverges.

Convergence Examples

Example: Determine the largest open disk $D[i, r]$ on which $\sum_{k=0}^{\infty} \frac{(z-i)^k}{2^k}$ converges.

Absolute Convergence

- **Definition:** The series $\sum_{k=0}^{\infty} c_k$ is called absolutely convergent if $\sum_{k=0}^{\infty} |c_k|$ converges.

- By the comparison test, taking $M_k = |c_k|$,

$$\sum_{k=0}^{\infty} |c_k| \ \text{ convergent} \implies \sum_{k=0}^{\infty} c_k \ \text{ convergent}$$

  That is, absolute convergence implies convergence.

Pointwise Convergence

- Consider a function $F_n(z)$ defined on a set T, where $F_n(z)$ depends on both a non-negative integer n and $z \in \mathbb{C}$.

  For example: $F_n(z) = \sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}$, and T is the disk $|z| < 1$.

- If for each $z \in \mathbb{C}$, $\lim_{n\to\infty} F_n(z)$ exists and equals $F(z)$, we say that F_n converges pointwise to F.

- **Definition:** F_n converges pointwise to F on T if for each $z \in T$, given $\epsilon > 0$ there is a natural number N (possibly depending on both ϵ and z) such that if $n > N$ then $|F_n(z) - F(z)| < \epsilon$.

Pointwise Convergence, Continued

- For example, we saw that for $F_n(z) = \sum_{k=0}^{n} z^k = \dfrac{1 - z^{n+1}}{1 - z}$, $F(z) = \dfrac{1}{1 - z}$, and again T is the disk $|z| < 1$.

- Notice: $|F_n(z) - F(z)| = \left| \dfrac{z^{n+1}}{1 - z} \right|$ depends on both n and z. In order to make this difference small, n must be chosen with reference to the particular z being considered.

- Here $F_n(z) \to F(z)$ pointwise on T

Uniform Convergence

- Again consider a function $F_n(z)$ defined on a set T, where $F_n(z)$ depends on both a non-negative integer n and $z \in \mathbb{C}$.

- **Definition:** F_n converges uniformly to F on T if given $\epsilon > 0$ there is a natural number N (possibly depending on ϵ but not on any particular z) such that if $n > N$ then for any $z \in T$, $|F_n(z) - F(z)| < \epsilon$.

- Roughly speaking, if $F_n \to F$ uniformly, for n large enough the difference $|F_n(z) - F(z)|$ will be small for every $z \in T$.

Uniform Convergence, Continued

- Again consider $F_n(z) = \sum_{k=0}^{n} z^k = \dfrac{1 - z^{n+1}}{1 - z}$ and $F(z) = \dfrac{1}{1-z}$, but this time let T be the disk $|z| < 1/2$.

- Again

$$|F_n(z) - F(z)| = \left| \frac{z^{n+1}}{1-z} \right| < \frac{(1/2)^{n+1}}{(1/2)} = \frac{1}{2^n}$$

- Notice: $|F_n(z) - F(z)|$ is bounded by an expression which is independent of z and which goes to zero as $n \to \infty$: $F_n \to F$ uniformly on T.

Taylor Series

Taylor Series Definition

- **Definition:** Suppose f is holomorphic at z_0. Then

$$\sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z - z_0)^j$$

is called the Taylor Series for f around z_0.

- If $z_0 = 0$ the series above is instead called a Maclaurin Series

Taylor Series Example

Example: Construct the Maclaurin series for $f(z) = e^z$

Solution: $f(0) = f'(0) = f''(0) = f'''(0) = \cdots = e^0 = 1$, so the Maclaurin series is

$$\sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} z^j = \sum_{j=0}^{\infty} \frac{1}{j!} z^j = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

The Main Result

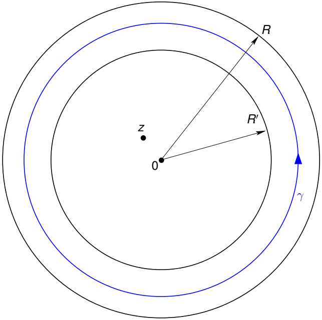
- Under what conditions is a function equal to its Taylor series?
- **Theorem:** If f is holomorphic in an open disk $D[z_0, R]$, then

$$f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z - z_0)^j$$

  for every $z \in D[z_0, R]$.

  Furthermore, the series converges uniformly in any closed subdisk $\overline{D}[z_0, R']$ where $R' < R$.
- Consequently, the Taylor series will converge to $f(z)$ everywhere inside the largest disk centred at z_0 over which $f(z)$ is holomorphic.

Proof in the case $z_0 = 0$



Let $\gamma = C[z_0, (R' + R)/2]$.

For any z in $\overline{D}[z_0, R']$,

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

$$= \frac{1}{2\pi i}\int_\gamma f(\zeta)\left[\frac{1}{\zeta}\cdot\frac{1}{1 - z/\zeta}\right] d\zeta \quad \}\ \text{notice } |z/\zeta| < 1$$

$$= \frac{1}{2\pi i}\int_\gamma f(\zeta)\left[\frac{1}{\zeta}\cdot\left(\sum_{j=0}^{n}(z/\zeta)^j + \frac{(z/\zeta)^{n+1}}{1 - z/\zeta}\right)\right] d\zeta$$

Proof in the case $z_0 = 0$, continued

Splitting this last expression:

$$\sum_{j=0}^{n} \frac{z^j}{j!} \left(\frac{j!}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta^{j+1}}\, d\zeta \right) + \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \cdot \left(\frac{z}{\zeta}\right)^{n+1} \, d\zeta$$

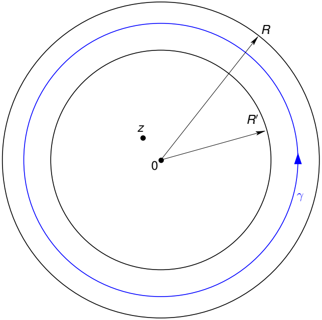
$$= \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!} z^j + \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \cdot \left(\frac{z}{\zeta}\right)^{n+1} \, d\zeta$$

Notice: as $n \to \infty$ the first sum becomes the desired Taylor series.

It remains to show that

$$\lim_{n\to\infty} \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \cdot \left(\frac{z}{\zeta}\right)^{n+1} \, d\zeta = 0$$

Proof in the case $z_0 = 0$, continued



$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\cdot\left(\frac{z}{\zeta}\right)^{n+1}\, d\zeta$$

On γ,

$$\left|\frac{1}{\zeta - z}\right| \le \frac{1}{\left(\frac{R+R'}{2} - R'\right)} = \frac{2}{R - R'}$$

and

$$\left|\frac{z}{\zeta}\right|^{n+1} = \frac{|z|^{n+1}}{|\zeta|^{n+1}} \le \left[\frac{R'}{\left(\frac{R'+R}{2}\right)}\right]^{n+1} = \left(\frac{2R'}{R'+R}\right)^{n+1} = \alpha^{n+1}$$

where $\alpha < 1$

Proof in the case $z_0 = 0$, continued

Using these bounds we have

$$\left| \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \cdot \left(\frac{z}{\zeta}\right)^{n+1} \, d\zeta \right|$$

$$\leq \quad \frac{1}{2\pi} \max_{\zeta \in \gamma} |f(\zeta)| \left(\frac{2}{R - R'}\right) \alpha^{n+1}$$

$$\to \quad 0 \ \text{ as } \ n \to \infty$$