Math 372 - Introductory Complex Variables

Mar 27 2019

Power and Laurent Series

Taylor Series Example

Example: Find the Taylor series about $z = 0$ (i.e. the Maclaurin series) for $\mathrm{Log}(1 - z)$ and determine the disk over which is it valid.

Important Taylor (Maclaurin) Series

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots, \quad \forall z \in \mathbb{C}$$

$$\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots, \quad \forall z \in \mathbb{C}$$

$$\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots, \quad \forall z \in \mathbb{C}$$

$$\operatorname{Log}(1+z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} z^k}{k} = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots, \quad |z| < 1$$

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k = 1 + z + z^2 + z^3 + \cdots, \quad |z| < 1$$

Power Series

- Power series about z_0: $\sum_{j=0}^{\infty} a_j (z - z_0)^j$
- **Theorem:** For each power series there is a real number $0 \leq R \leq \infty$ called the radius of convergence such that the series
  - converges for $|z - z_0| < R$
  - converges uniformly for $|z - z_0| \leq R' < R$
  - diverges for $|z - z_0| > R$

Power Series continued

- As a consequence of the uniform convergence, $f(z) = \sum_{j=0}^{\infty} a_j(z - z_0)^j$ defines a holomorphic function on the disk $D[z_0, R]$
- Furthermore, if $f(z) = \sum_{j=0}^{\infty} a_j(z - z_0)^j$, then $a_j = \frac{f^{(j)}(z_0)}{j!}$.

Power Series and Uniform Convergence

- Letting $f_n(z) = \sum_{j=0}^{n} a_j(z - z_0)^j$, a power series is really just $\lim_{n\to\infty} f_n(z)$, and the results on power series follow from:
- **Theorem:** If each f_n is continuous on a region G, $n = 1, 2, \ldots$, and $f_n \to f$ uniformly on G, then f is continuous on G.
- **Theorem:** If f_n is continuous on G, $n = 1, 2, \ldots$, and $f_n \to f$ uniformly on G, then $\int_\gamma f_n(z)\, dz \to \int_\gamma f(z)\, dz$ for every γ in G.
- **Theorem:** If each f_n is holomorphic in a region G in which all closed piecewise simple smooth paths are G-contractible, $n = 1, 2, \ldots$, and $f_n \to f$ uniformly on G, then f is holomorphic on G.

Operations with Power Series

- **Theorem:** A power series can be integrated and differentiated termwise within its radius of convergence.

- **Theorem:** Suppose $f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j$ and $g(z) = \sum_{j=0}^{\infty} b_j (z - z_0)^j$ define holomorphic functions about z_0, then

  (i) $cf(z) = \sum_{j=0}^{\infty} ca_j (z - z_0)^j$ where c is a constant

  (ii) $(f + g)(z) = \sum_{j=0}^{\infty} (a_j + b_j)(z - z_0)^j$

Operations with Power Series, continued

Theorem: If $f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j$ and $g(z) = \sum_{j=0}^{\infty} b_j (z - z_0)^j$ define holomorphic function about z_0, then fg is holomorphic at z_0 and

$$(fg)(z) = \sum_{j=0}^{\infty} c_j (z - z_0)^j$$

where

$$c_j = \sum_{k=0}^{j} a_{j-k} b_j$$

Example

Example: Find the Taylor series about $z = 0$ (i.e. the Maclaurin series) for $f(z) = e^{-z^2}$ and state the radius of convergence.

Example

Example: Find the Taylor series about $z = 0$ (i.e. the Maclaurin series) for $f(z) = \dfrac{z}{(1-z)^2}$ and state the radius of convergence.

Laurent Series

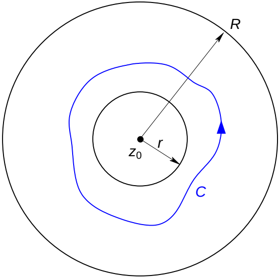
- **Definition:** A point z_0 is a singularity of f if f is not holomorphic at z_0 but z_0 is the limit of a sequence of points at which f is holomorphic.

- For example, $f(z) = \frac{e^z}{z-i}$ has a singularity at $z = i$.

- Can we find a Taylor-series-like representation of a function about its singularities?

Laurent Series, continued

Theorem: Suppose f is holomorphic on the annulus (washer shaped region) $r < |z - z_0| < R$:



Then f can be expressed as

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} a_{-j} (z - z_0)^{-j} \quad (1)$$

$$= \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j, \quad \text{where...} \quad (2)$$

Laurent Series, continued

- the series (1) converges on $r < |z - z_0| < R$
- convergence is uniform on $r < \rho_1 \leq |z - z_0| \leq \rho_2 < R$, and
- the coefficients a_j are given by

$$a_j = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{j+1}}\, d\zeta$$

where C is any positively oriented simple closed contour lying inside the annulus and containing z_0.

Laurent Series, continued

Furthermore, if for $r < R$ we have series such that

- $\sum_{j=0}^{\infty} a_j(z - z_0)^j$ converges for $|z - z_0| < R$, and
- $\sum_{j=1}^{\infty} a_{-j}(z - z_0)^{-j}$ converges for $|z - z_0| > r$

then

$$f(z) = \sum_{j=-\infty}^{\infty} a_j(z - z_0)^j$$

defines a holomorphic function on $r < |z - z_0| < R$ with

$$a_j = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} \, d\zeta$$

Laurent Series, continued

- A Laurent series can often be constructed using known series, as opposed to resorting to contour integrals for determining the coefficients.

- For this purpose, it is useful to recall the geometric series for $|z| < 1$:

$$\frac{1}{1-z} = \sum_{j=0}^{\infty} z^j$$