Math 372 - Introductory Complex Variables

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The Residue Theorem

An Important Integral

Recall:

Suppose G is a region, γ is a simple closed piecewise smooth positively oriented G-contractible contour, z_0 is inside γ, and n is an integer. Then

$$\int_\gamma (z - z_0)^n \, dz = \begin{cases} 2\pi i \ \text{ if } n = -1 \\ 0 \ \text{ if } n \neq -1 \end{cases}$$

Application to Laurent Series

Again for G a region and γ a simple closed piecewise smooth positively oriented G-contractible contour, suppose f is holomorphic in G except at the single isolated singularity z_0 inside γ.

Then by Laurent's theorem there is some punctured disk $D = \{z \in \mathbb{C} \ : \ 0 < |z - z_0| < R\}$ inside γ on which

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j$$

continued...

Application to Laurent Series

Let γ_0 be a circle with center z_0 sufficiently small to be contained in D, say $\gamma_0 = C[z_0, R/2]$.Then

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \int_{\gamma_0} f(z)\,dz \quad \text{(by Cauchy's Theorem)} \\
&= \int_{\gamma_0} \sum_{j=-\infty}^{\infty} a_j (z-z_0)^j\,dz \\
&= \sum_{j=-\infty}^{\infty} \int_{\gamma_0} a_j (z-z_0)^j\,dz \\
&= a_{-1} 2\pi i
\end{aligned}$$

Residues

- **Definition:** If f has an isolated singularity at z_0 then the coefficient a_{-1} of $(z - z_0)^{-1}$ in the Laurent series expansion for f about z_0 is called the residue of f at z_0, and denoted $\mathrm{Res}(f; z_0)$.
- **Example:**

$$f(z) = z^3 \exp(1/z) = z^3 + z^2 + \frac{z}{2!} + \frac{1}{3!} + \frac{1}{4!z} + \frac{1}{5!z^2} + \cdots$$

  about the isolated singularity at $z = 0$. So $\mathrm{Res}(f; 0) = 1/4!$
- Using this result with, say, $\gamma = C[0, 1]$, the positively oriented unit circle:

$$\int_\gamma z^3 e^{1/z}\, dz = 2\pi i [\mathrm{Res}(f; 0)] = \frac{2\pi i}{4!} = \frac{\pi i}{12}$$

Finding Residues

- As previous example shows, one way to find residues of f is to simply work out the Laurent series.
- If z_0 is a removable singularity then the Laurent series contains only non-negative powers of $(z - z_0)$, so $\text{Res}(f; z_0) = a_{-1} = 0$
- If z_0 is a simple pole, then

$$f(z) = \frac{a_{-1}}{(z - z_0)} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

so

$$(z - z_0)f(z) = a_{-1} + a_0(z - z_0) + a_1(z - z_0)^2 + \cdots$$

so

$$\lim_{z \to z_0}(z - z_0)f(z) = a_{-1}$$

Finding Residues, continued

If z_0 is pole of order 2, then

$$f(z) = \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

so

$$(z-z_0)^2 f(z) = a_{-2} + a_{-1}(z-z_0) + a_0(z-z_0)^2 + a_1(z-z_0)^3 + \cdots$$

so

$$\frac{d}{dz}\left[(z-z_0)^2 f(z)\right] = a_{-1} + 2a_0(z-z_0) + 3a_1(z-z_0)^2 + \cdots$$

so

$$\lim_{z\to z_0} \frac{d}{dz}\left[(z-z_0)^2 f(z)\right] = a_{-1}$$

Finding Residues, continued

If z_0 is pole of order 3, then

$$f(z) = \frac{a_{-3}}{(z-z_0)^3} + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots$$

so

$$(z-z_0)^3 f(z) = a_{-3} + a_{-2}(z-z_0) + a_{-1}(z-z_0)^2 + a_0(z-z_0)^3 + \cdots$$

so

$$\frac{d^2}{dz^2}\left[(z-z_0)^3 f(z)\right] = 2 \cdot a_{-1} + 3 \cdot 2 \cdot a_0(z-z_0) + 4 \cdot 3 \cdot a_1(z-z_0)^2 + \cdots$$

so

$$\lim_{z \to z_0} \frac{1}{2!} \frac{d^2}{dz^2}\left[(z-z_0)^3 f(z)\right] = a_{-1}$$

Finding Residues, continued

Theorem: If f has a pole of order m at z_0, then

$$\operatorname{Res}(f; z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)]$$

Putting it All Together: The Residue Theorem

- Suppose G is a region, γ is a simple closed piecewise smooth positively oriented G-contractible contour, and that f is holomorphic in G except at the isolated singularities $z_1, z_2, \ldots, z_n$ inside γ. We wish to evaluate

$$\int_\gamma f(z)\,dz$$

- Letting $\gamma_1, \gamma_2, \ldots, \gamma_n$ be sufficiently small circles with centres $z_1, z_2, \ldots, z_n$, respectively, we saw previously that by deforming γ we have

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \cdots + \int_{\gamma_n} f(z)\,dz$$

continued...

Putting it All Together: The Residue Theorem

- But

$$\int_{\gamma_j} f(z)\, dz = 2\pi i \cdot \mathrm{Res}(f; z_j)$$

- So

$$\int_{\gamma} f(z)\, dz = 2\pi i \cdot \mathrm{Res}(f; z_1) + 2\pi i \cdot \mathrm{Res}(f; z_2) + \cdots + 2\pi i \cdot \mathrm{Res}(f; z_n)$$

Cauchy's Residue Theorem

Theorem: Suppose G is a region, γ is a simple closed piecewise smooth positively oriented G-contractible contour, and f is holomorphic in G except at the isolated singularities $z_1, z_2, \ldots, z_n$ inside γ. Then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{n} \mathrm{Res}(f; z_j)$$