Continuity and the Derivative

Jan 28 2019

Limits

Recall...

- **Definition:** Let $f : G \to \mathbb{C}$ and z_0 be an accumulation point of G. Then
$$\lim_{z \to z_0} f(z) = w_0$$
if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $0 < |z - z_0| < \delta$ then $|f(z) - w_0| < \epsilon$.

- Limit laws for sums, differences, products, powers similar to those for real functions.

Continuous Functions

- **Definition:** Let $f : G \to \mathbb{C}$ and $z_0 \in G$. If z_0 is an isolated point of G or if
$$\lim_{z \to z_0} f(z) = f(z_0)$$
then f is continuous at z_0.

- That is, f is continuous at z_0 if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $0 < |z - z_0| < \delta$ then $|f(z) - f(z_0)| < \epsilon$.

- f is continuous on a set G if it is continuous at every $z_0 \in G$.

Continuous Functions, continued

- By the limit laws, sums, differences, products and quotients of continuous functions are again continuous on their domains. In particular...

- **Theorem:**
  1. Polynomial functions

     $$p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}, \quad a_i \in \mathbb{C}, \quad n \in \mathbb{N}$$

     are continuous on $\mathbb{C}$.

  2. Rational functions $R(z) = \dfrac{p(z)}{q(z)}$ where $p(z)$ and $q(z)$ are polynomials are continuous at each $z_0 \in \mathbb{C}$ provided $q(z_0) \neq 0$.

Continuous Functions, continued

Theorem: (Continuous functions of continuous functions are continuous)

Suppose $g : G \to \mathbb{C}$ and $f : H \to \mathbb{C}$ are both continuous on their domains, and that $g(G) \subset H$. If $z_0 \in G$ with $\lim_{z \to z_0} g(z) = w_0 \in H$, then $\lim_{z \to z_0} f(g(z)) = f(w_0)$. That is,

$$\lim_{z \to z_0} f(g(z)) = f(\lim_{z \to z_0} g(z))$$

Continuous Functions, continued

Proof: Let $\epsilon > 0$ be given.

Since $\lim_{w \to w_0} f(w) = f(w_0)$ by the continuity of f, there is $\delta_1 > 0$ such that $0 < |w - w_0| < \delta_1 \implies |f(w) - f(w_0)| < \epsilon$.

Since $\lim_{z \to z_0} g(z) = w_0$, for $\delta_1 > 0$ there is a $\delta_2 > 0$ $0 < |z - z_0| < \delta_2 \implies |g(z) - w_0| < \delta_1$.

Take $\delta = \delta_2$. Then if $0 < |z - z_0| < \delta$, $|g(z) - w_0| < \delta_1$, so that

$$|f(g(z)) - f(w_0)| = |f(g(z)) - f(g(z_0))| < \epsilon$$

The Derivative

- **Definition:** Suppose $f : G \to \mathbb{C}$ and z_0 is an interior point of G. The derivative of f at z_0 is

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

  if this limits exists. If so, f is said to be differentiable at z_0 .

- If f is differentiable at each point of some $D[z_0, r] \subset G$ then f is holomorphic at z_0.

The Derivative, continued

- A function differentiable (resp. holomorphic) on a set is differentiable (resp. holomorphic) at each point of the set.

- A function holomorphic on $\mathbb{C}$ is called entire.

- The definition of the derivative can instead be expressed

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

  where $h \in \mathbb{C}$.

Derivative Rules

Theorem: Let f and g be differentiable at z, h differentiable at $g(z)$, and $a, b \in \mathbb{C}$. Then

1. $(af(z) + bg(z))' = af'(z) + bg'(z)$
2. $(f(z)g(z))' = f'(z)g(z) + f(z)g'(z)$
3. $\left(\frac{f(z)}{g(z)}\right)' = \frac{g(z)f'(z) - f(z)g'(z)}{(g(z))^2}$
4. $(h(g(z)))' = h'(g(z) \bullet g'(z)$
5. $(z^n)' = nz^{n-1}$ for $n \in \mathbb{Z} \setminus \{0\}$

Proof of Power Rule for $n \geq 1$

Let $f(z) = z^n$. If $n = 1$ the result is trivial. For $n \geq 2$,

$$
\begin{aligned}
(z^n)' &= \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \\
&= \lim_{h \to 0} \frac{(z+h)^n - z^n}{h} \\
&= \lim_{h \to 0} \frac{z^n + nz^{n-1}h + (\text{terms with a factor of } h^2) + \cdots - z^n}{h} \\
&= \lim_{h \to 0} nz^{n-1} + (\text{terms with factor of } h) \\
&= nz^{n-1}
\end{aligned}
$$