Topology of $\mathbb{C}$

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Topology

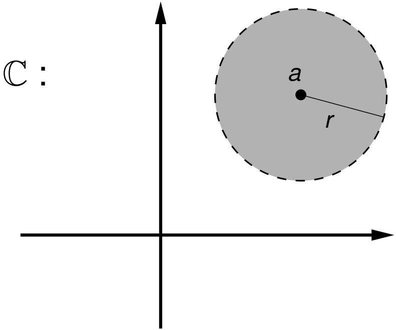
Wikipedia: Topology is concerned with the properties of space that are preserved under continuous deformations, such as stretching, twisting, crumpling and bending, but not tearing or gluing.

Open Disks

Definition: Let $a \in \mathbb{C}$ and $r > 0$ be real. The set

$$D[a, r] = \{z \in \mathbb{C} : |z - a| < r\}$$

is called the open disk of radius r and centre a.

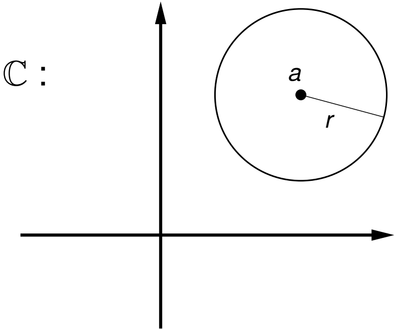


Circles

Definition: Let $a \in \mathbb{C}$ and $r > 0$ be real. The set

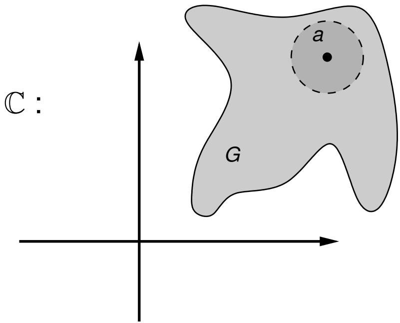
$$C[a, r] = \{z \in \mathbb{C} : |z - a| = r\}$$

is the circle of radius r and centre a.



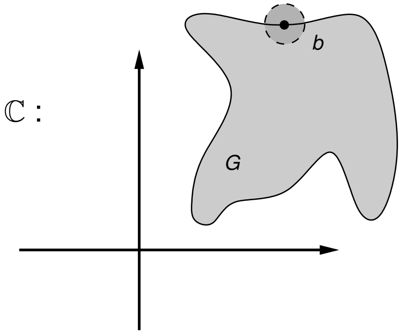
Interior Point

Definition: Let $G \subset \mathbb{C}$. $a \in G$ is an interior point of G if there is some open disk with centre a which is a subset of G. That is, there is some $r > 0$ such that $D[a, r] \subset G$



Boundary Point

Definition: Let $G \subset \mathbb{C}$. $b \in \mathbb{C}$ is a boundary point of G if every open disk with centre b contains a point in G and a point that is not in G. Notice: a boundary point of G need not be in G.



The set of all boundary points of a set G is denoted ∂G .

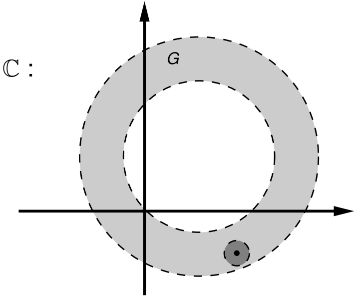
Accumulation Point and Isolated Point

Definition: Let $G \subset \mathbb{C}$. $c \in \mathbb{C}$ is an accumulation point of G if every open disk with centre c contains a point of G different from c.

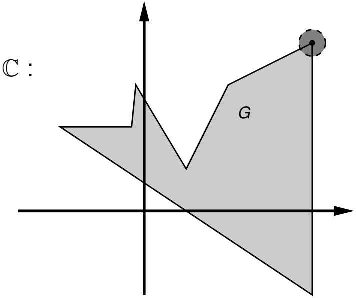
Definition: Let $G \subset \mathbb{C}$. $d \in \mathbb{C}$ is an isolated point of G if some open disk with centre d contains no other points of G.

Open Sets

Definition: A set $G \subset \mathbb{C}$ is open if every point of G is an interior point.



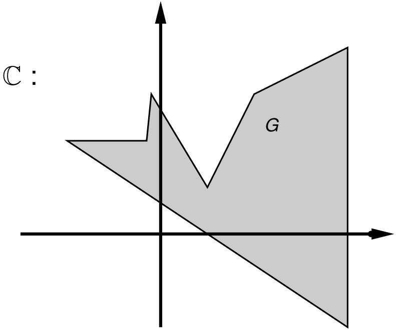
G is open



G is not open

Closed Sets

Definition: A set $G \subset \mathbb{C}$ is closed if it contains all of its boundary points.



Definition: The closure of a set G is $\overline{G} = G \cup \partial G$, the set together with all of its boundary points.

Bounded Sets

Definition: A set $G \subset \mathbb{C}$ is bounded if there is some $r > 0$ such that $G \subset D[0, r]$.

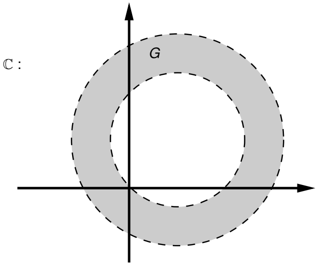
Example

Let S be the set of complex numbers which satisfy $1 < (\text{Im}(z))^2 < 4$.

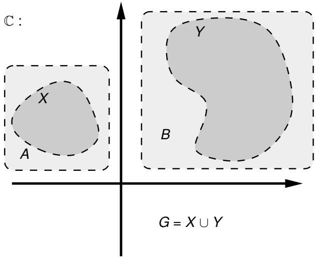
1. Is S open?
2. Is S bounded?
3. Describe the boundary points of S.
4. What is the closure of S?

Connected Set

Definition: Set $X, Y \subset \mathbb{C}$ are separated if there are disjoint open sets $A, B \subset \mathbb{C}$ such that $X \subset A$ and $Y \subset B$. A set $G \subset \mathbb{C}$ is connected if it cannot be expressed as a union of two nonempty separated sets.



G is connected



X and Y are each connected, but $G = X \cup Y$ is not

Definition: A region is a connected open set.

Paths

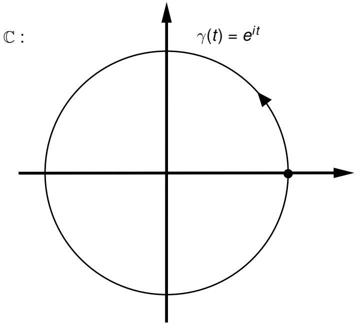
- **Definition:** A path or curve is a continuous function

$$\gamma : [a, b] \to \mathbb{C}$$

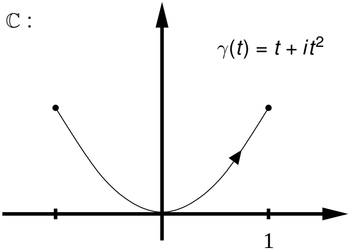
- A path γ is smooth if it is differentiable on $[a, b]$ and the derivative is continuous, nonzero, and both $\lim_{t \to a^+} \gamma'(t)$ and $\lim_{t \to b^-} \gamma'(t)$ exist.

- A path γ is simple if it is one-to-one. A path is simple and closed if it is one-to-one with the exception that $\gamma(a) = \gamma(b)$.

Examples of Paths



simple smooth closed path
$\gamma(t) = e^{it}, t \in [0, 2\pi]$



simple smooth path
$\gamma(t) = t + it^2, t \in [-1, 1]$

Notice the arrows on the paths indicating the orientation: the direction along the path corresponding to increasing t.

Paths and Connectedness

Theorem: $G \subset \mathbb{C}$ is connected if any two points in G can be connected by a path lying entirely in G.