Powers and Roots

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Periodicity of the complex exponential

- For $k \in \mathbb{Z}$,

$$e^{i(\theta+2k\pi)} = e^{i\theta} e^{i2k\pi} = e^{i\theta} \cdot 1 = e^{i\theta}$$

- Say that $e^{i\theta}$ is periodic with period 2π

Powers

- For $n \in \mathbb{N}$ it is easy to define z^n:

$$z = |z|e^{i\theta}$$

so

$$z^n = |z|^n e^{in\theta}$$

- In fact, true for $n \in \mathbb{Z}$ if $z^{-n} = 1/z^n$.

- Value of z^n is the same regardless of choice of arg(z) used to specify θ:

$$|z|^n e^{in(\theta+2k\pi)} = |z|^n e^{in\theta} e^{in2k\pi} = |z|^n e^{in\theta} \cdot 1$$

Roots

- For roots of complex numbers there is more to consider.
- **Definition:** For $m \in \mathbb{N}$, ζ is an m^{th} root of z if $\zeta^m = z$
- To find all m^{th} roots of a complex number $z = |z|e^{i\theta} \neq 0$, let $\zeta = \rho e^{i\phi}$ where $\rho > 0$.
- Then we must have $\rho^m e^{im\phi} = |z|e^{i\theta}$
- So $\rho = \sqrt[m]{|z|}$ and $e^{im\phi} = e^{i\theta}$

continued...

Roots, continued

- So $m\phi = \theta + 2k\pi, \quad \text{where} \quad k \in \mathbb{Z}$
- So $\phi = \dfrac{\theta}{m} + \dfrac{2k\pi}{m}, \quad \text{where} \quad k \in \mathbb{Z}$
- So all possible m^{th} roots of z are given by

$$\zeta = \sqrt[m]{|z|}e^{i(\theta+2k\pi)/m}, \quad k \in \mathbb{Z}$$

continued...

Roots, continued

- Notice: for $k = 0, 1, \ldots, m-1$ we have $0 \leq \dfrac{2k\pi}{m} < 2\pi$

- So
$$\zeta = \sqrt[m]{|z|}e^{i(\theta+2k\pi)/m}, \quad k = 0, 1, \ldots, m-1$$
represents m distinct m^{th} roots of z.

- Are these m roots the only ones? That is, what if $k \leq -1$ or $k \geq m$?

continued...

Roots, continued

- By the Division Algorithm there are integers q and r such that $k = qm + r$ where $0 \leq r \leq m - 1$
- So

$$\begin{aligned}
\zeta &= \sqrt[m]{|z|}e^{i(\theta+2k\pi)/m} \\
&= \sqrt[m]{|z|}e^{i(\theta+2(qm+r)\pi)/m} \\
&= \sqrt[m]{|z|}e^{i(\theta+2r\pi)/m}e^{i2qm\pi/m} \\
&= \sqrt[m]{|z|}e^{i(\theta+2r\pi)/m}
\end{aligned}$$

which, since $0 \leq r \leq m - 1$, is one of the roots we found already.

continued...

Roots, Conclusion

- **Theorem:** Let $m \geq 1$ be an integer and $z = re^{i\theta}$ with r, $\theta \in \mathbb{R}$, and where θ is any valid choice of arg(z). The m^{th} roots of z are given by

$$\boxed{z^{1/m} = \sqrt[m]{|z|}e^{i(\theta+2k\pi)/m}, \quad k = 0, 1, \ldots, m-1}$$

- **Corollary:** If m and n are positive integers with no common factors, then $(z^{1/n})^m = (z^m)^{1/n}$ and this common number, denoted by $z^{m/n}$ is given by

$$z^{m/n} = \sqrt[n]{|z|^m}e^{im(\theta+2k\pi)/n}, \quad k = 0, 1, \ldots, n-1$$

Roots of Unity

- **Definition:** Let $m \in \mathbb{N}$. if $\zeta \in \mathbb{C}$ has the property that $\zeta^m = 1$ then ζ is called an m^{th} root of unity. If $\zeta^m = 1$ but $\zeta^k \neq 1$ for $k = 1, 2, \ldots, m-1$ then ζ is said to be a primitive m^{th} root of unity.

- So the m^{th} roots of unity are

$$1^{1/m} = \sqrt[m]{|1|}e^{i(0+2k\pi)/m}, \quad k = 0, 1, \ldots, m-1$$

which reduces to

$$\boxed{1^{1/m} = e^{i2k\pi/m}, \quad k = 0, 1, \ldots, m-1}$$

Roots of Unity

Letting $\omega_m = e^{i2\pi/m}$, notice that $\left\{1, \omega_m^1, \omega_m^2, \ldots, \omega_m^{m-1}\right\}$ is the complete set of m^{th} root of unity.