Polar Form and the Exponential Function

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An Observation about $|z|$, $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$

Theorem: For $z = a + bi \in \mathbb{C}$,

$$-|z| \leq \mathrm{Re}(z) \leq |z| \quad \text{and} \quad -|z| \leq \mathrm{Im}(z) \leq |z|$$

Proof:

$$-\sqrt{a^2 + b^2} \leq a \leq \sqrt{a^2 + b^2}$$

and

$$-\sqrt{a^2 + b^2} \leq b \leq \sqrt{a^2 + b^2}$$

Sums and Differences of Conjugates

Theorem: For $z = a + bi \in \mathbb{C}$,

$$\boxed{\mathrm{Re}(z) = \frac{z + \overline{z}}{2} \quad \text{and} \quad \mathrm{Im}(z) = \frac{z - \overline{z}}{2i}}$$

Proof:

$$z + \overline{z} = a + bi + \overline{(a + bi)} = a + bi + a - bi = 2a = 2\mathrm{Re}(z) \ .$$

Similarly

$$z - \overline{z} = a + bi - \overline{(a + bi)} = a + bi - (a - bi) = 2bi = 2i\,\mathrm{Im}(z) \ .$$

The Triangle Inequality

Theorem: For $z, w \in \mathbb{C}$, $|z + w| \leq |z| + |w|$

Proof:

$$
\begin{aligned}
|z+w|^2 &= (z+w)\overline{(z+w)} \\
&= (z+w)(\bar{z}+\overline{w}) \\
&= z\bar{z} + z\overline{w} + \bar{z}w + w\overline{w} \\
&= z\bar{z} + w\overline{w} + z\overline{w} + \overline{z\overline{w}} \\
&= z\bar{z} + w\overline{w} + 2\mathrm{Re}(z\overline{w}) \\
&\leq |z|^2 + |w|^2 + 2|z\overline{w}| \\
&= |z|^2 + |w|^2 + 2|z||w| \\
&= (|z| + |w|)^2
\end{aligned}
$$

Taking square roots: $\boxed{|z + w| \leq |z| + |w|}$

The Reverse Triangle Inequality

Theorem: For $z, w \in \mathbb{C}$, $||z| - |w|| \leq |z + w|$

Proof:

$|z| = |z + w - w| \leq |z + w| + |w|$ by the triangle inequality,

so

$$|z| - |w| \leq |z + w| \; .$$

Similary

$$|w| - |z| \leq |z + w| \; .$$

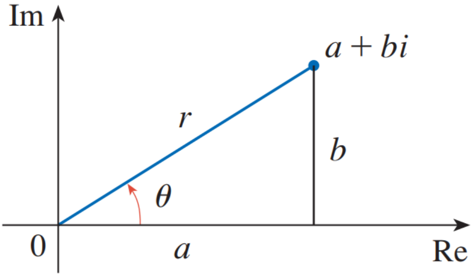
One of $|z| - |w|$ and $|w| - |z|$ is non-negative so is equal to $||z| - |w||$, so

$$\boxed{||z| - |w|| \leq |z + w|}$$

Polar Form of a Complex Number

For $z = a + bi$, convert corresponding point (a, b) to polar coordinates:

$$a = r\cos\theta, \quad b = r\sin\theta$$



So $\boxed{z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)}$

Polar Form of a Complex Number

- For $z = a + bi = r\cos\theta + ir\sin\theta$,

$$r = \sqrt{a^2 + b^2} \text{ and } \tan\theta = \frac{b}{a}$$

- $z = r(\cos\theta + i\sin\theta)$ sometimes written $z = r\,\text{cis}(\theta)$

- θ is called an argument of z, written $\theta = \arg(z)$.

- If θ is an argument of z, so are $\theta + 2k\pi$ for any $k \in \mathbb{Z}$.

The Real Exponential Function

- Recall: for $x \in \mathbb{R}$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

- This means for each $x \in \mathbb{R}$

$$e^x = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{x^k}{k!}$$

The Complex Exponential Function

- We'll see later: for $z \in \mathbb{C}$, $\displaystyle\sum_{k=0}^{\infty} \frac{z^k}{k!}$ also converges.

- Define

$$\boxed{e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{z^k}{k!}}$$

- With this definition:

  (1) $e^z e^w = e^{z+w}$

  (2) $\dfrac{e^z}{e^w} = e^{z-w}$

  (3) $(e^z)^n = e^{nz}$ for $n \in \mathbb{Z}$

The Complex Exponential Function, continued

- Letting $z = i\theta$ where $\theta \in \mathbb{R}$:

$$\begin{aligned} e^{i\theta} &= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \\ &= (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots) + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots) \\ &= \cos\theta + i \sin\theta \end{aligned}$$

- Euler's equation: $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$
- As a consequence

$$z = \underbrace{r\left(\cos\theta + i\sin\theta\right)}_{\text{polar form}} = \overbrace{re^{i\theta}}^{\text{exponential form}}$$

The Most Beautiful Theorem in Mathematics

Letting $z = i\pi$ in Euler's equation:

$$e^{i\pi} = \cos\pi + i\sin\pi,$$

that is,

$$e^{i\pi} = -1$$

or

$$e^{i\pi} + 1 = 0$$

DeMoivre's Formula

For $n \in \mathbb{N}$ and $\theta \in \mathbb{R}$, since

$$\left(re^{i\theta}\right)^n = r^n e^{in\theta} = r^n[\cos(n\theta) + i\sin(n\theta)],$$

we have DeMoivre's formula

$$\boxed{z^n = r^n[\cos(n\theta) + i\sin(n\theta)]}$$