Definition and Basic Properties of the Complex Numbers

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The Complex Numbers $\mathbb{C}$

- **Definition:** The set of complex numbers is

  $$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R} \text{ and } i^2 = -1\}$$

  We define $\sqrt{-1} = i$.

- Say that $a + bi = c + di$ iff (if and only if) $a = c$ and $b = d$.

- For $z = a + bi$, $\text{Re}(z) = a$ is called the real part of z and $\text{Im}(z) = b$ is the imaginary part. If $a = 0$, so $z = bi$, z is said to be pure imaginary.

The Complex Numbers $\mathbb{C}$, continued

- Addition and subtraction:

$$(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i$$

- Multiplication:

$$(a+bi)(c+di) = ac+adi+bci+bdi^2 = (ac-bd)+(ad+bc)i$$

- Division: for $c + di \neq 0$,

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \left(\frac{ac+bd}{c^2+d^2}\right) + \left(\frac{bc-ad}{c^2+d^2}\right) i$$

- These definitions, along with $0 = 0 + 0i$ and $1 = 1 + 0i$ make $(\mathbb{C}, +, \bullet)$ a field

Fields

Definition: A field is a set F together with two operations $+$ and $\bullet$ such that for any $x, y, z \in F$:

1. $x + y \in F$
2. $x + y = y + x$
3. $(x + y) + z = x + (y + z)$
4. There exists a zero element $0 \in F$ such that $0 + x = x$
5. There exists an element $-x$ such that $x + (-x) = 0$
6. $x \bullet y \in F$
7. $x \bullet y = y \bullet x$
8. $(x \bullet y) \bullet z = x \bullet (y \bullet z)$
9. There exists a unit element $1 \in F$ such that $1 \bullet x = x$
10. If $x \neq 0$ there exists an element $1/x$ such that $(1/x) \bullet x = 1$
11. $x \bullet (y + z) = x \bullet y + x \bullet z$
12. $1 \neq 0$

Examples of Fields

- Familiar: $(\mathbb{Q}, +, \bullet)$ and $(\mathbb{R}, +, \bullet)$ are both fields

- More unusual: Recall that for $a, p \in \mathbb{N}$,

  $a \bmod p =$ remainder upon division of a by p.

  Let p be a prime number and $\mathbb{F} = \{0, 1, 2, \ldots, p-1\}$.

  For $a, b \in \mathbb{F}$ define

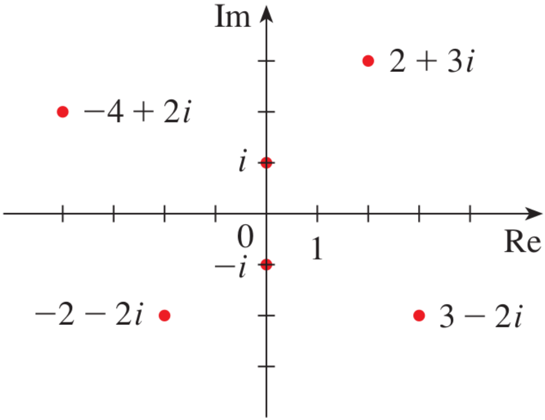
  $$a +_{\mathbb{F}} b = (a + b) \bmod p \quad \text{and} \quad a \bullet_{\mathbb{F}} b = ab \bmod p$$

  Then $(\mathbb{F}, +_{\mathbb{F}}, \bullet_{\mathbb{F}})$ is a field

Definition of $\mathbb{C}$ revisited

- $z = a + bi$ consists of a real part a and an imaginary part b: it is a 2 dimensional object
- $a + bi$ can instead be represented as a pair $(a, b) \in \mathbb{R}^2$
- Addition then becomes $(a, b) + (c, d) = (a + c, b + d)$
- Multiplication becomes $(a, b)(c, d) = (ac - bd, ad + bc)$
- This defines a field structure on $\mathbb{R}^2$ and gives us a way to visualize or geometrically represent $\mathbb{C}$, just as we use the 1-dimensional number line to represent $\mathbb{R}$.

The Complex or Argand Plane



Notice: this allows us to interpret complex numbers as vectors.

Absolute Value

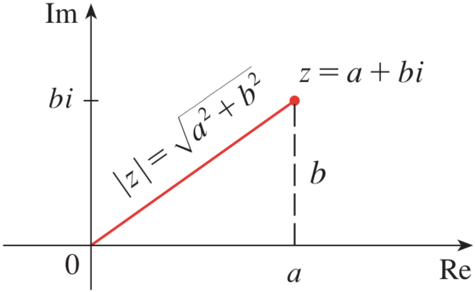
- **Definition:** The absolute value or modulus of $z = a + bi$ is

$$|z| = |a + bi| = \sqrt{a^2 + b^2}$$

- $|z| =$ the distance from z to 0

- $|z - w| = 0$ iff $z = w$

- $|z| = 0$ iff $z = 0$



Complex Conjugates

- **Definition:** The complex conjugate of $z = a + ib$ is $\overline{z} = a - bi$
- $\overline{z + w} = \overline{z} + \overline{w}$
- $\overline{zw} = \overline{z}\,\overline{w}$
- $\overline{\left(\frac{z}{w}\right)} = \frac{(\overline{z})}{(\overline{w})}$
- $\overline{(z^n)} = (\overline{z})^n$ for $n \in \mathbb{N}$
- $z\overline{z} = |z|^2$