Question 1: Solve

$$y'' + 4y' + 5y = 35e^{-4x}, \quad y(0) = -3, \quad y'(0) = 1$$

using the method of undetermined coefficients to find y_p.

$\underline{y_c}$: $r^2+4r+5=0$

$$r = \frac{-4 \pm \sqrt{16-4(1)(5)}}{2}$$

$$= \frac{-4 \pm 2i}{2}$$

$$= -2 \pm i$$

$\therefore\ y_c = c_1 e^{-2x}\cos(x) + c_2 e^{-2x}\sin(x).$

$\underline{y_p}$: Try $y_p = Ae^{-4x} \Rightarrow y_p' = -4Ae^{-4x},\ y_p'' = 16Ae^{-4x}.$

$\therefore\ 16Ae^{-4x} + 4[-4Ae^{-4x}] + 5[Ae^{-4x}] = 35e^{-4x}$

$\Rightarrow\ 5A = 35$

$\Rightarrow\ A = 7.$

$\therefore\ y = y_c + y_p$

$= c_1 e^{-2x}\cos(x) + c_2 e^{-2x}\sin(x) + 7e^{-4x}$

$y(0) = -3 \Rightarrow c_1 + 7 = -3 \Rightarrow c_1 = -10$

$y'(0) = 1 \Rightarrow -10\left[-2e^{-2x}\cos(x) - e^{-2x}\sin(x)\right]_{x=0} + c_2\left[-2e^{-2x}\sin(x) + e^{-2x}\cos(x)\right]_{x=0}$

$-28e^{-4x}\Big|_{x=0} = 1$

$\Rightarrow\ 20 + c_2 - 28 = 1$

$\Rightarrow\ c_2 = 9$

$\therefore\ y = -10e^{-2x}\cos(x) + 9e^{-2x}\sin(x) + 7e^{-4x}$

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Question 2: Find the general solution to on $(0,\infty)$

$$x^2y'' - 4xy' = x^5 \quad \Big\} \text{ Cauchy Euler}$$

using variation of parameters to find y_p.

y_c: $x^2y'' - 4xy' = 0$

Let $y = x^m$, $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

$\Rightarrow x^2 m(m-1)x^{m-2} - 4x\, m x^{m-1} = 0$

$\Rightarrow (m^2 - m - 4m)x^m = 0$

$\Rightarrow m(m-5) = 0$

$\Rightarrow m = 0, \; m = 5$

$\therefore y_c = c_1 \cdot 1 + c_2 x^5 \quad \Big\} \; W = \begin{vmatrix} 1 & x^5 \\ 0 & 5x^4 \end{vmatrix} = 5x^4$

y_p: $x^2y'' - 4xy' = x^5$

$\Rightarrow y'' - \frac{4}{x}y' = \underbrace{x^3}_{f(x)}$

$u_1 = \int \frac{-y_2 f(x)}{W}\,dx = \int -\frac{x^5 \cdot x^3}{5x^4}\,dx = -\frac{1}{25}x^5$

$u_2 = \int \frac{y_1 f(x)}{W}\,dx = \int \frac{1 \cdot x^3}{5x^4}\,dx = \frac{1}{5}\ln(x)$

$\therefore y_p = u_1y_1 + u_2y_2 = \left(-\frac{1}{25}x^5\right)(1) + \left(\frac{1}{5}\ln(x)\right)(x^5)$

$= \underbrace{-\frac{1}{25}x^5}_{\text{part of } y_c \text{, so omit.}} + \frac{1}{5}x^5\ln(x)$

$$\boxed{\therefore y = c_1 + c_2x^5 + \frac{1}{5}x^5\ln(x)}$$

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Question 3: A mass of 1 kg stretches a spring $g/2$ m where $g = 9.8$ m/s^2 is the usual gravitational constant. The stretched spring with 1 kg mass attached is allowed to come to rest at equilibrium. The mass travels through a medium which imparts a damping force equal to 3 times the velocity, and starting at time $t = 0$ s an external force of $f(t) = 2\sin(t)$ is applied to the system. Find the equation of motion for the system for times $t \geq 0$.

(You may use any method you like to solve the resulting differential equation.)

Spring constant: $-1g = -k\frac{g}{2} \Rightarrow k = 2.$

Equation is $1 \cdot x'' + 3x' + 2x = 2\sin(t)$, $x(0) = x'(0) = 0$.

x_c: $r^2 + 3r + 2 = 0 \Rightarrow (r+2)(r+1) = 0 \Rightarrow r = -2, -1$

$\therefore x_c = C_1 e^{-2t} + C_2 e^{-t}$.

x_p: $x_p = A\sin(t) + B\cos(t)$

$x_p' = A\cos(t) - B\sin(t)$

$x_p'' = -A\sin(t) - B\cos(t)$

$$[-A\sin(t) - B\cos(t)] + 3[A\cos(t) - B\sin(t)] + 2[A\sin(t) + B\cos(t)] = 2\sin(t)$$

$$\Rightarrow \left.\begin{array}{l} A - 3B = 2 \\ 3A + B = 0 \end{array}\right\} \quad \left.\begin{array}{l} B = -3A \\ A - 3(-3A) = 2 \end{array}\right\} \quad \therefore \begin{array}{l} A = \frac{1}{5} \\ B = -\frac{3}{5} \end{array}$$

$$\therefore x(t) = C_1 e^{-2t} + C_2 e^{-t} + \frac{1}{5}\sin(t) - \frac{3}{5}\cos(t)$$

$x(0) = 0 \Rightarrow C_1 + C_2 = \frac{3}{5}$

$+ \qquad +$

$x'(0) = 0 \Rightarrow -2C_1 - C_2 = -\frac{1}{5}$

$-C_1 = \frac{2}{5} \Rightarrow C_1 = \frac{-2}{5} \Rightarrow C_2 = \frac{3}{5} - \left(\frac{-2}{5}\right) = 1$

$$\therefore x(t) = \left(-\frac{2}{5}\right)e^{-2t} + e^{-t} + \frac{1}{5}\sin(t) - \frac{3}{5}\cos(t)$$

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Question 4: Solve using the Laplace transform:

$$y'' + y = \sqrt{2}\sin(\sqrt{2}t), \quad y(0) = 10, \quad y'(0) = 0$$

$$\mathcal{L}\{y''+y\} = \mathcal{L}\{\sqrt{2}\sin(\sqrt{2}t)\}$$

$$s^2Y - sy(0) - \cancel{y'(0)}^{0} + Y = \sqrt{2}\ \frac{\sqrt{2}}{s^2+2}$$

$$Y(s^2+1) - 10s = \frac{2}{s^2+2}$$

$$Y = \underbrace{\frac{2}{(s^2+2)(s^2+1)}}_{\#} + 10\frac{s}{s^2+1}$$

$$Y = 2\underbrace{\left[\frac{1}{(s^2+(\sqrt{2})^2)(s^2+1^2)}\right]}_{\#32} + 10\underbrace{\left[\frac{s}{s^2+1^2}\right]}_{8}$$

$$\therefore y = 2\left[\frac{\sqrt{2}\sin(t) - \sin(\sqrt{2}t)}{\sqrt{2}(2-1)}\right] + 10\cos(t)$$

$$= \boxed{2\sin(t) - \sqrt{2}\sin(\sqrt{2}t) + 10\cos(t)}$$

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Question 5: Solve using the Laplace transform:

$$y' + y = f(t), \quad \text{where } y(0) = -1 \text{ and } f(t) = \begin{cases} 0, 0 \le t < 2 \\ e^t, t \ge 2 \end{cases}$$

Here $f(t) = e^t u(t-2)$

$$\mathcal{L}\{y' + y\} = \mathcal{L}\{e^t u(t-2)\}$$

$$sY - y(0) + Y = e^{-2s} \mathcal{L}\{e^{t+2}\}$$

$$(s+1)Y + 1 = e^{-2s} e^{2} \frac{1}{s-1}$$

$$\therefore Y = e^2 e^{-2s} \frac{1}{(s-1)(s+1)} - \frac{1}{s+1}$$

$$\therefore y(t) = e^2 \left[\frac{e^t - e^{-t}}{1-(-1)}\right]_{t \to t-2} u(t-2) - e^{-t}$$

$$= e^2 \left[\frac{e^{t-2} - e^{-t+2}}{2}\right] u(t-2) - e^{-t}$$

$$= \boxed{\left(\frac{e^t - e^{4-t}}{2}\right) u(t-2) - e^{-t}}$$

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