Question 1: For this question use the autonomous differential equation

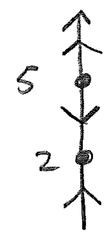
$$\frac{dy}{dt} = (y-2)(e^{y-5} - 1)$$

(i) Determine the equilibrium solutions

$\frac{dy}{dt} = 0$ at $\boxed{y=2}$ and $\boxed{y=5}$

[2]

(ii) Sketch the one dimensional phase portrait.



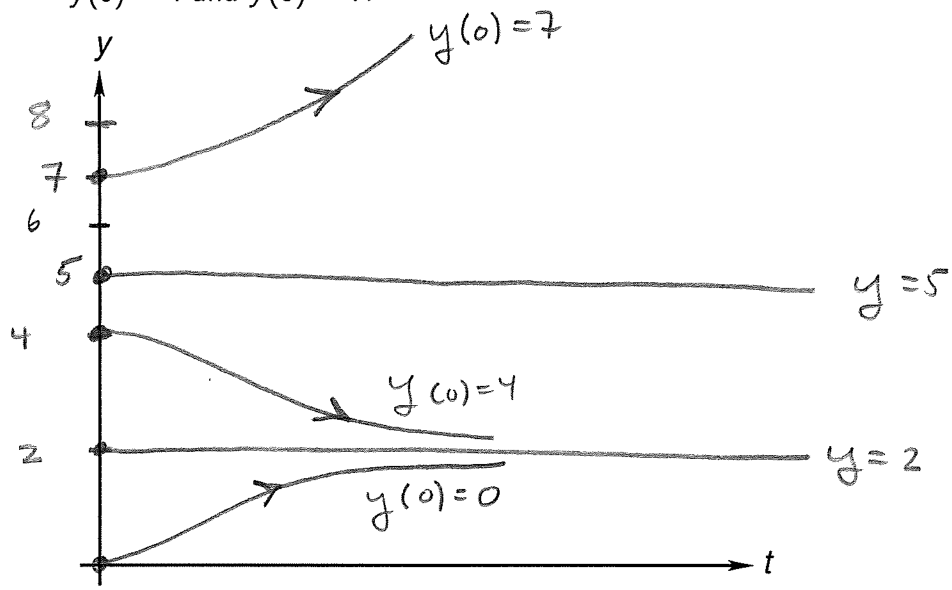
[2]

(iii) Classify each equilibrium solution as stable, unstable or semi-stable

y=5 is unstable, y=2 is stable.

[2]

(iv) Sketch the equilibrium solution curves as well as approximate solution curves corresponding to $y(0) = 0$, $y(0) = 4$ and $y(0) = 7$:



[2]

(v) Use Euler's method with $h = 0.1$ and $y(0) = 4$ to approximate $y(0.2)$ to three decimal places.

$f(t,y) = (y-2)(e^{y-5}-1)$

$y_{n+1} = y_n + h(y_n-2)(e^{y_n-5}-1)$

$h=0.1$

n	t_n	y_n	y_{n+1}
0	0	4	3.8736
1	0.1	3.8736	3.7470
2	0.2	3.7470	$\boxed{3.622}$

[2]

Question 2: Solve the IVP $\dfrac{dy}{dx} = \dfrac{x(x^2+1)}{4y^3}$, $y(0) = \dfrac{-1}{\sqrt{2}}$. State your final answer in explicit form.

$$\int 4y^3\,dy = \int (x^3 + x)\,dx$$

$$y^4 = \frac{x^4}{4} + \frac{x^2}{2} + C$$

$$y = \cancel{\pm} - \left[\frac{x^4}{4} + \frac{x^2}{2} + C\right]^{\frac{1}{4}}$$

$y(0) = \dfrac{-1}{\sqrt{2}}$:

note!

$$-\frac{1}{\sqrt{2}} = -\left[\frac{0^4}{4} + \frac{0^2}{2} + C\right]^{\frac{1}{4}}$$

so $C = \frac{1}{4}$

$$\boxed{\therefore\ y = -\left[\frac{x^4}{4} + \frac{x^2}{2} + \frac{1}{4}\right]^{\frac{1}{4}}}$$

[5]

Question 3: Solve the IVP $(x+1)\dfrac{dy}{dx} + y = \ln(x)$, $y(1) = 10$, and give the largest interval $\mathcal{I}$ over which the solution is defined. State your final answer in explicit form.

$$\frac{dy}{dx} + \frac{1}{(x+1)}\,y = \frac{\ln(x)}{1+x}$$

$P(x) = \dfrac{1}{x+1}$, $f(x) = \dfrac{\ln(x)}{1+x}$,

both continuous on $(0,\infty)$.

$$\mu(x) = e^{\int P(x)\,dx} = e^{\int \frac{1}{x+1}\,dx} = e^{\ln(1+x)} = 1+x$$

$$\therefore\ y = \frac{\int \mu(x) f(x)\,dx}{\mu(x)}$$

$$= \frac{\int (1+x)\frac{\ln(x)}{(1+x)}\,dx}{1+x}$$

$$= \frac{x\ln(x) - x + C}{1+x}$$

$y(1) = 10$:

$$\frac{\cancel{1\ln(1)}^{\to 0} - 1 + C}{1+1} = 10$$

$$C = (2)(10) + 1 = 21$$

$$\boxed{\therefore\ y = \frac{x\ln(x) - x + 21}{1+x},\quad \mathcal{I} = (0,\infty)}$$

[5]

Question 4: Find the general solution to $x\dfrac{dy}{dx} = 2xe^x - y + 6x^2$. You may leave your solution in implicit form.

$\underbrace{(2xe^x - y + 6x^2)}_{M}dx + \underbrace{(-x)}_{N}dy = 0$

$M_y = -1 = N_x$, so exact.

$f(x,y) = \int N dy$

$= \int (-x) dy$

$= -xy + g(x)$.

$f_x(x,y) = M$

$\Rightarrow -y + g'(x) = 2xe^x - y + 6x^2$

$\Rightarrow g(x) = \int (2xe^x + 6x^2) dx$

$= 2xe^x - 2e^x + 2x^3 + C$

$\therefore f(x,y) = -xy + 2e^x(x-1) + 2x^3 + C$

General solution is

$\boxed{-xy + 2e^x(x-1) + 2x^3 + C = 0.}$

[5]

Question 5: Find the general solution to $(y^2 + yx)dx - x^2dy = 0$ and also make note of any constant solutions. You may leave your solution in implicit form.

Equation can be written

$\dfrac{dy}{dx} = \dfrac{y^2 + yx}{x^2}$, so $\boxed{y=0}$

is a constant solution.

$(y^2 + yx)dx - x^2dy = 0 \;\}\, *$

is homogeneous of degree 2.

Let $y = ux$, $dy = udx + xdu$,

so $(*)$ becomes

$(u^2x^2 + ux^2)dx - x^2(udx + xdu) = 0$

$u^2x^2dx - x^3du = 0$

separable:

$\int \frac{1}{x} dx = \int \frac{1}{u^2} du$

$\ln|x| = \frac{-1}{u} + C$

$\boxed{\ln|x| = -\frac{x}{y} + C}$

or

$\boxed{x + y(\ln|x| - c) = 0}$

or

$\boxed{y = \dfrac{x}{c - \ln|x|}}$

[5]

Question 6: Solve the IVP: $x^2\frac{dy}{dx} - 2xy = 3y^4$, $y(1) = 1/2$. You may leave your solution in implicit form.

$\left.\frac{dy}{dx} - \frac{2}{x}y = \frac{3}{x^2}y^4\right\}$ Bernoulli, $n=4$

Let $u = y^{1-4} = y^{-3}$,

so $y = u^{-\frac{1}{3}}$, $\frac{dy}{dx} = -\frac{1}{3}u^{-4/3}\frac{du}{dx}$:

$-\frac{1}{3}u^{-4/3}\frac{du}{dx} - \frac{2}{x}u^{-1/3} = \frac{3}{x^2}u^{-4/3}$

$\Rightarrow \frac{du}{dx} + \frac{6}{x}u = \frac{-9}{x^2}$

$\mu(x) = e^{\int \frac{6}{x}dx} = x^6$

so $u = \dfrac{\int x^6 \cdot \left(\frac{-9}{x^2}\right)dx}{x^6}$

$= \dfrac{\left(-\frac{9}{5}x^5 + C\right)}{x^6}$

so $y^{-3} = \frac{-9}{5x} + \frac{C}{x^6}$.

$y(1) = \frac{1}{2}$, so

$\left(\frac{1}{2}\right)^{-3} = \frac{-9}{5} + C \Rightarrow C = \frac{49}{5}$.

$\boxed{\therefore\ y^{-3} = \frac{-9}{5x} + \frac{49}{5x^6}}$

[5]

Question 7: Solve the IVP: $\frac{dy}{dx} = \frac{1-x-y}{x+y}$, $y(0) = -2$. State your final answer in explicit form.

Let $u = x+y$

so $\frac{du}{dx} = 1 + \frac{dy}{dx}$.

Equation becomes

$\frac{du}{dx} - 1 = \frac{1-u}{u}$

$\frac{du}{dx} - 1 = \frac{1}{u} - 1$

$\frac{du}{dx} = \frac{1}{u}$

$\int u\,du = \int dx$

$\frac{u^2}{2} = x + C_1$

$u = \pm\sqrt{2x + C_2}$

$x + y = \pm\sqrt{2x + C_2}$

$y = -x \pm \sqrt{2x + C_2}$.

$y(0) = -2$:

$-2 = 0 \pm \sqrt{0 + C_2}$

$\therefore C_2 = 4$

So

$\boxed{y = -x - \sqrt{2x+4}}$

[5]