Note: Structure your proofs by stating the full "**Proposition:**" followed by "**Proof:**" as in class, and please start each new proof on a new page. Indent your argument where appropriate for readability and be careful with notation.

As with any proof, clarity of presentation is as important as solving the problem. Strive to make your proofs clear, concise and precise. Feel free to use results from earlier problems in subsequent ones.

Finally, proofs for many of the problems in this course are likely available online if one looks hard enough, but the point of this course is to learn analysis, and in particular how to write analysis proofs. As such, please resist turning to the internet for help except as a last resort; instead, discuss solutions among your peers or see me for hints.

The first two (short) questions deal with some facts that we have used previously, so we should probably prove them:

1. Let p be a prime number and $n \in \mathbb{N}$ with $n \geq 2$. Prove that $p^{1/n}$ is irrational.
2. Let $u \in \mathbb{R} \setminus \mathbb{Q}$ and suppose that $\{a, b, c\} \subset \mathbb{Q}$ with $ac \neq 0$. Prove that $\dfrac{au + b}{c} \in \mathbb{R} \setminus \mathbb{Q}$.
3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at c and that for every open interval (a, b) containing c there is $x \in (a, b)$ such that $x \neq c$ and $f(x) = f(c)$. Determine the value of $f'(c)$ and prove your result.
4. Prove the Product Rule: If $f : (a, b) \to \mathbb{R}$ and $g : (a, b) \to \mathbb{R}$ are both differentiable at $c \in (a, b)$ then so is $h = fg$ with
$$h'(c) = f(c)g'(c) + f'(c)g(c)$$
5. Let $c \in \mathbb{R}$ be fixed and define $f : \mathbb{R} \to \mathbb{R}$ by
$$f(x) = \begin{cases} c + x^3 & \text{if } x \in \mathbb{Q} \\ c & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$
Find $f'(0)$ and prove your result.
6. Suppose that $g : (a, b) \to \mathbb{R}$ is differentiable and that $g' : (a, b) \to \mathbb{R}$ is bounded. Show that g must also be bounded. (Hint: Mean Value Theorem.)
7. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function with the property that there exists an $\alpha > 1$ such that $|f(x) - f(y)| \leq |x - y|^{\alpha}$ for every x and y. Show that f must be a constant function.