Note: Structure your proofs by stating the full "**Proposition:**" followed by "**Proof:**" as in class, and please start each new proof on a new page. Indent your argument where appropriate for readability and be careful with notation.

As with any proof, clarity of presentation is as important as solving the problem. Strive to make your proofs clear, concise and precise. Feel free to use results from earlier problems in subsequent ones.

Finally, proofs for many of the problems in this course are likely available online if one looks hard enough, but the point of this course is to learn analysis, and in particular how to write analysis proofs. As such, please resist turning to the internet for help except as a last resort; instead, discuss solutions among your peers or see me for hints.

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1. Prove that $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^2$ is continuous at $x = -1$ using the $\epsilon\delta$ definition of continuity.
2. Prove that $f : (0, \infty) \to \mathbb{R} : x \mapsto 1/x$ is continuous using the $\epsilon\delta$ definition of continuity. (Hint: let $c > 0$ be arbitrary but fixed and show that f is continuous at c.)
3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and that $f(c) > 0$. Show that there is some $\alpha > 0$ such that $f(x) > 0$ for every $x \in (c - \alpha, c + \alpha)$.
4. Find an example of a bounded discontinuous function $f : [0, 1] \to \mathbb{R}$ that has neither an absolute maximum nor absolute minimum. ~~Show that~~ Explain why your function has the required properties.
5. Show that if $f : [a, b] \to \mathbb{R}$ is continuous then $f([a, b])$ is either a closed and bounded interval or a single real number.
6. Consider the function $f : [0, 1] \to \mathbb{R}$ defined as follows: let $\{r_i\}_{i=1}^{\infty}$ be an enumeration of the rational numbers in the interval $(0, 1)$, and define
$$f(x) = \sum_{r_i \le x} \frac{1}{2^i} \ .$$
Here's another way to think about this definition: for $x \in [0, 1]$, let $S = \{i \in \mathbb{N} \ : \ r_i \le x\}$. Then
$$f(x) = \sum_{i \in S} \frac{1}{2^i} \ ,$$
and if $S = \emptyset$ we define
$$f(x) = \sum_{i \in \emptyset} \frac{1}{2^i} = 0 \ .$$
In either case, f is the function that jumps by $1/2^i$ at each rational r_i.
   (a) What is $f(0)$?
   (b) What is $f(1)$?
   (c) Show that f is discontinuous at every rational in $(0, 1)$. Hint: for any rational r in $(0, 1)$ consider a sequence of rational numbers $\{x_n\}_{n=1}^{\infty}$ that increases to r. Show that $f(x_n) \not\to f(r)$.
