Note: Structure your proofs by stating the full "**Proposition:**" followed by "**Proof:**" as in class, and please start each new proof on a new page. Indent your argument where appropriate for readability and be careful with notation.

As with any proof, clarity of presentation is as important as solving the problem. Strive to make your proofs clear, concise and precise. Feel free to use results from earlier problems in subsequent ones.

Finally, proofs for many of the problems in this course are likely available online if one looks hard enough, but the point of this course is to learn analysis, and in particular how to write analysis proofs. As such, please resist turning to the internet for help except as a last resort; instead, discuss solutions among your peers or see me for hints.

The first two problems deal with intervals of $\mathbb{R}$. We didn't formally define intervals in class, but the meaning (as a set) is the same as that used in calculus. See the beginning of Section 1.4 of the text for the formal definition. **Caution:** we did not define the notion of the limit of a union (or intersection), so don't be tempted to apply it.

1. Prove that
   (a) Every closed interval can be expressed as the intersection of a countable number of open intervals. That is, if $[a, b] \subset \mathbb{R}$, then there exist $(a_i, b_i) \subset \mathbb{R}$, $i \in \mathbb{N}$, such that
   $$[a, b] = \bigcap_{i=1}^{\infty} (a_i, b_i)$$
   (b) Every open interval can be expressed as the union of a countable number of closed intervals. That is, if $(a, b) \subset \mathbb{R}$, then there exist $[a_i, b_i] \subset \mathbb{R}$, $i \in \mathbb{N}$, such that
   $$(a, b) = \bigcup_{i=1}^{\infty} [a_i, b_i]$$
2. Show that if S is a set of disjoint open intervals in $\mathbb{R}$ then S is countable.
3. Find $\lim\limits_{n\to\infty} 2^{-n}$ and prove your result. Do not make use of logarithms in your proof; use only techniques and results developed in the course so far.
4. Find $\lim\limits_{n\to\infty} \dfrac{\sqrt{n}}{n^2+1}$ and prove your result.
5. Suppose that $S \subset \mathbb{R}$ and that S is nonempty and bounded. Show that there exists a monotone increasing sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in S$ for each $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} x_n = \sup(S)$.
6. Prove that there exists a sequence $\{x_n\}_{n=1}^{\infty}$ with the property that for each $y \in \mathbb{R}$ there is a subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ with $x_{n_i} \to y$.
7. Suppose $\{x_n\}_{n=1}^{\infty}$ is bounded. Show that $\lim_{n\to\infty} x_n = x$ if and only if every subsequence of $\{x_n\}_{n=1}^{\infty}$ converges to x.