Question 1:

(a) [3 points] Evaluate $f'(0)$ where $f(x) = x \arccos(x) - \sqrt{1-x^2}$.

$$f'(x) = \arccos(x) + x\left(\frac{-1}{\sqrt{1-x^2}}\right) - \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)$$

$$f'(0) = \arccos(0)$$

$$= \boxed{\frac{\pi}{2}}$$

(b) [3 points] Evaluate $g'(0)$ where $g(x) = \cosh(x)\sinh(x^2)$.

$$g'(x) = \sinh(x)\sinh(x^2) + \cosh(x)\cosh(x^2)(2x)$$

$$\therefore \quad g'(0) = \boxed{0}$$

(c) [4 points] Evaluate

$$\lim_{x \to 0+} \sin(x)\ln(x) \quad \sim \text{"}0 \cdot (-\infty)\text{"}$$

$$= \lim_{x \to 0^+} \frac{\ln x}{\left(\frac{1}{\sin(x)}\right)} \sim \text{"}\frac{-\infty}{\infty}\text{"}$$

$$\stackrel{H}{=} \lim_{x \to 0^+} \frac{\frac{1}{x}}{\left(\frac{-1}{\sin^2 x}\right)\cos x}$$

$$= \lim_{x \to 0^+} -\frac{\sin^2 x}{x\cos x} \sim \text{"}\frac{0}{0}\text{"}$$

$$\stackrel{H}{=} \lim_{x \to 0^+} \frac{-2\sin x\cos x}{\cos x - x\sin x}$$

$$= \boxed{0}$$

Question 2:

(a) [4 points] Evaluate

$$\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{te^t}\right) \quad \sim \quad \text{"}\pm(\infty-\infty)\text{"}$$

$$= \lim_{t\to 0} \frac{e^t-1}{te^t} \quad \sim \text{"}\frac{0}{0}\text{"}$$

$$\overset{H}{=} \lim_{t\to 0} \frac{\cancel{e^t}}{\cancel{e^t}+t\cancel{e^t}}$$

$$= \lim_{t\to 0} \frac{1}{1+t}$$

$$= \boxed{1}$$

(b) [3 points] An animal gains mass at a rate of $W(t) = \dfrac{100t}{(t^2+1)^3}$ kg/yr, where t is time in years. What is the total mass gain during the first two years of life?

Let $w(t)$ = mass at time t years, so $w'(t) = W(t)$.

$$w(2)-w(0) = \int_0^2 \frac{100t}{(t^2+1)^3}\,dt \qquad \begin{cases} u = t^2+1, \ du = 2t\,dt \\ t=0 \Rightarrow u=1 \\ t=2 \Rightarrow u=5 \end{cases}$$

$$= 50\int_1^5 u^{-3}\,du$$

$$= \frac{50}{-2}\left[u^{-2}\right]_1^5$$

$$= -25\left[\frac{1}{25}-1\right] = \boxed{24 \text{ kg}}$$

(c) [3 points] The average value of $f(x) = qx^2$ over the interval $[-q, q]$ is 9. Determine the value of the constant q.

$$9 = \frac{1}{q-(-q)}\int_{-q}^{q} qx^2\,dx$$

$$= \frac{1}{2\cancel{q}}\cdot\cancel{q}\cdot\left[\frac{x^3}{3}\right]_{-q}^{q}$$

$$\therefore \quad 18 = \frac{q^3}{3}-\left(\frac{-q^3}{3}\right)$$

$$18 = \frac{2}{3}q^3$$

$$\therefore \quad q = \left[\frac{(3)(18)}{2}\right]^{1/3} = \boxed{3}$$

Question 3:

(a) **[3 points]** Define the function

$$F(x) = \int_{-1}^{x^3} \frac{\cos(t^2)}{e^t}\,dt$$

Evaluate and simplify $F(-1) - F'(0)$.

$$F(-1) = \int_{-1}^{(-1)^3} \frac{\cos(t^2)}{e^t}\,dt = \int_{-1}^{-1} \frac{\cos(t^2)}{e^t}\,dt = 0$$

$$F'(x) = \frac{\cos((x^3)^2)}{e^{x^3}} \cdot 3x^2$$

$\cdot$ $F'(0) = 0$

$$\therefore \quad F(-1) - F'(0) = \boxed{0}$$

(b) **[3 points]** Evaluate:

$$I = \int \frac{1}{x \ln x}\,dx$$

let $u = \ln x$, $du = \frac{1}{x}\,dx$

$$\therefore \quad I = \int \frac{1}{u}\,du$$

$$= \ln|u| + C$$

$$= \boxed{\ln|\ln x| + C}$$

(c) **[4 points]** Evaluate:

$$\int_0^1 (\sqrt[4]{x} + 1)^2\,dx$$

$$= \int_0^1 x^{\frac{1}{2}} + 2x^{\frac{1}{4}} + 1\,dx$$

$$= \left[\frac{2}{3}x^{3/2} + 2\left(\frac{4}{5}\right)x^{\frac{5}{4}} + x\right]_0^1$$

$$= \frac{2}{3} + \frac{8}{5} + 1$$

$$= \frac{10 + 24 + 15}{15}$$

$$= \boxed{\frac{49}{15}}$$

Question 4:

(a) [3 points] Evaluate:

$$I = \int \frac{x+2}{(x-1)^{3/2}}\,dx$$

let $u = x-1$
$du = dx$

$$\therefore\ I = \int \frac{u+3}{u^{3/2}}\,du$$

$$= \int u^{-\frac{1}{2}} + 3u^{-\frac{3}{2}}\,du$$

$$= 2u^{\frac{1}{2}} - 3(2)\,u^{-\frac{1}{2}} + C$$

$$= \boxed{2(x-1)^{\frac{1}{2}} - 6(x-1)^{-\frac{1}{2}} + C}$$

(b) [4 points] Evaluate the following limit of Riemann Sums by interpreting it as $\int_a^b f(x)\,dx$ for some a, b and $f(x)$:

$$I = \lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^3 + \left(\frac{2}{n}\right)^3 + \cdots + \left(\frac{n}{n}\right)^3\right]$$

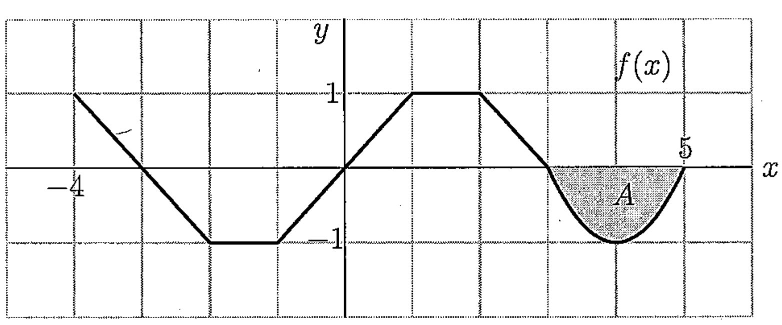
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \left(\frac{1}{n}\right)$$

$$\therefore\ x_i = \frac{i}{n}, \quad \Delta x = \frac{1}{n}, \quad f(x_i) = (x_i)^3$$

$$\therefore\ I = \int_0^1 x^3\,dx$$

$$= \left[\frac{x^4}{4}\right]_0^1 = \boxed{\frac{1}{4}}$$

(c) [3 points] For the function $f(x)$ whose graph is shown below, $\int_{-4}^{5} f(x)\,dx = \frac{-5}{6}$. Determine the area of the shaded region A.



$$\therefore\ \int_{-4}^{-3} f(x)\,dx + \int_{-3}^{3} f(x)\,dx + \int_{3}^{5} f(x)\,dx = \frac{-5}{6}$$

$$\frac{1}{2} + 0 - A = \frac{-5}{6}$$

$$\therefore\ A = \frac{1}{2} + \frac{5}{6} = \frac{3+5}{6} = \boxed{\frac{4}{3}}$$

Question 5 [8 points]: Evaluate:

$$I = \int x^3(\ln x)^2\,dx$$

$$u = (\ln x)^2 \quad ; \quad dv = x^3$$
$$du = 2\ln x\,\frac{1}{x} \quad ; \quad v = \frac{x^4}{4}$$

$$\therefore\ I = (\ln x)^2\,\frac{x^4}{4} - \int \frac{x^4}{4}\cdot 2\ln x\cdot\frac{1}{x}\,dx$$

$$= \frac{(\ln x)^2 x^4}{4} - \frac{1}{2}\underbrace{\int x^3 \ln x\,dx}$$

$$u = \ln x \quad ; \quad dv = x^3$$
$$du = \frac{1}{x}\,dx \quad ; \quad v = \frac{x^4}{4}$$

$$= \frac{(\ln x)^2 x^4}{4} - \frac{1}{2}\left[(\ln x)\left(\frac{x^4}{4}\right) - \int \frac{x^4}{4}\,\frac{1}{x}\,dx\right]$$

$$= \frac{(\ln x)^2 x^4}{4} - \frac{1}{8}(\ln x)\,x^4 + \frac{1}{8}\int x^3\,dx$$

$$= \boxed{\frac{(\ln x)^2 x^4}{4} - \frac{1}{8}(\ln x)\,x^4 + \frac{1}{32}\,x^4 + C}$$

Question 6 [8 points]: Evaluate:

$$I = \int \frac{5x-8}{x^2+x-12}\,dx$$

$$\frac{5x-8}{x^2+x-12} = \frac{5x-8}{(x+4)(x-3)} = \frac{A}{x+4} + \frac{B}{x-3}$$

$$= \frac{A(x-3)+B(x+4)}{(x+4)(x-3)}$$

$$= \frac{(A+B)x \; -3A+4B}{(x+4)(x-3)}.$$

$$\therefore \; A+B = 5 \;\Rightarrow\; B = 5-A$$

$$-3A+4B = -8 \;\Rightarrow\; -3A+4(5-A) = -8$$

$$-7A+20 = -8$$

$$-7A = -28$$

$$\therefore \; A = 4$$

$$\therefore \; B = 5-4 = 1$$

$$\therefore \; I = \int \frac{4}{x+4} + \frac{1}{x-3}\,dx$$

$$= \boxed{4\ln|x+4| + \ln|x-3| + C}$$

Question 7 [8 points]: Evaluate:

$$I = \int \sqrt{16 - x^2}\, dx$$

$$x = 4\sin\theta$$

$$dx = 4\cos\theta\, d\theta$$

$$\therefore I = \int \sqrt{16 - 16\sin^2\theta}\; 4\cos\theta\, d\theta$$

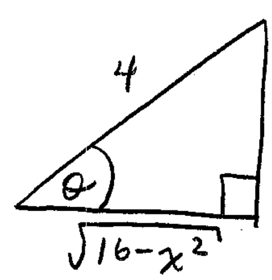
$$= \int 4\sqrt{1-\sin^2\theta}\; 4\cos\theta\, d\theta$$

$$= 16\int \cos^2\theta\, d\theta$$

$$= 16\int \frac{1}{2} + \frac{\cos(2\theta)}{2}\, d\theta$$

$$= 16\left[\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right] + C$$

$$= 8\theta + 8\sin\theta\cos\theta + C$$



$$\therefore \sin\theta = \frac{x}{4}$$

$$\cos\theta = \frac{\sqrt{16-x^2}}{4}$$

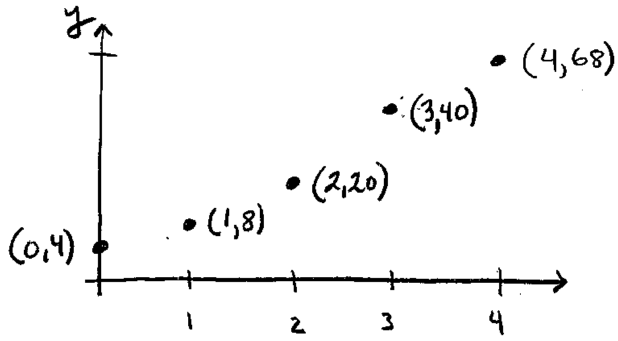
$$\theta = \arcsin\left(\frac{x}{4}\right)$$

$$\therefore I = 8\arcsin\left(\frac{x}{4}\right) + 8\left(\frac{x}{4}\right)\frac{\sqrt{16-x^2}}{4} + C$$

$$= \boxed{8\arcsin\left(\frac{x}{4}\right) + \frac{1}{2}\cdot x\cdot\sqrt{16-x^2} + C}$$

Question 8:

(a) [4 points] Consider the integral $\int_0^4 4(1+x^2)\,dx$. Let M_2 be the midpoint rule approximation of the integral using two subintervals, and T_2 the trapezoid rule approximation using two subintervals. Determine $T_2 - M_2$.



$\Delta x = \frac{4-0}{2} = 2$

$T_2 = \left(\frac{4+20}{2}\right)(2) + \left(\frac{20+68}{2}\right)(2) = 112$

$M_2 = (8)(2) + (40)(2) = 96.$

$\therefore\ T_2 - M_2 = 112 - 96 = \boxed{16}$

(b) [4 points] Determine the error in T_2, the trapezoid rule approximation used in part (a). Recall, the error in using the trapezoid rule to approximate $\int_a^b f(x)\,dx$ is at most $\frac{K(b-a)^3}{12n^2}$, where $|f''(x)| \le K$ on $[a,b]$.

$f(x) = 4 + 4x^2$

$f'(x) = 8x$

$f''(x) = 8 = K.$

$\therefore\ |E_{T_2}| \leq \frac{8(4-0)^3}{12 \cdot 2^2} = \frac{8 \cdot \cancel{4} \cdot \cancel{4} \cdot 4}{3 \cdot \cancel{4} \cdot \cancel{4}} = \boxed{\frac{32}{3}}$

Question 9:

(a) **[4 points]** Evaluate the improper integral

$$\int_0^\infty \frac{x^3}{e^{(x^4)}}\,dx$$

$$= \lim_{b\to\infty} \int_0^b x^3 e^{-x^4}\,dx$$

$$= \lim_{b\to\infty} \left[\frac{e^{-x^4}}{-4}\right]_0^b$$

$$= \lim_{b\to\infty} \left[\frac{e^{-b^4}}{-4} + \frac{1}{4}\right]$$

$$= \boxed{\frac{1}{4}}$$

(b) **[4 points]** Evaluate the improper integral:

$$\int_2^3 \frac{1}{\sqrt{3-x}}\,dx$$

$$= \lim_{b\to 3^-} \int_2^b [3-x]^{-\frac{1}{2}}\,dx$$

$$= \lim_{b\to 3^-} \left[-2[3-x]^{\frac{1}{2}}\right]_2^b$$

$$= \lim_{b\to 3^-} \left[-2(3-b)^{\frac{1}{2}} + 2(3-2)^{\frac{1}{2}}\right]$$

$$= \boxed{2}$$

(c) **[2 points]** Use the Comparison Theorem to show that the following integral diverges:

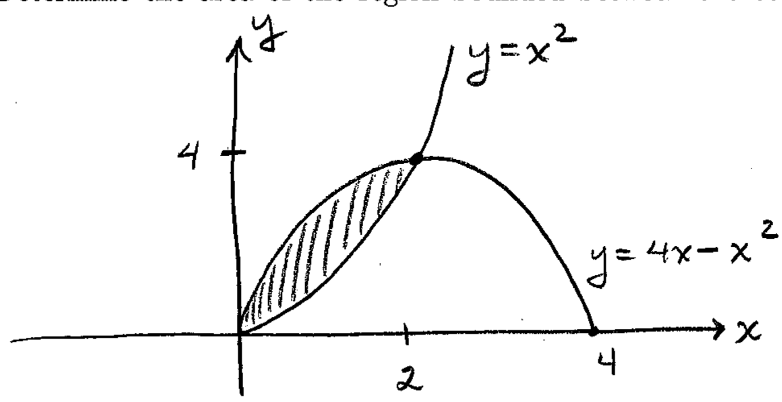
$$\int_1^\infty \frac{x^3+\sin^2 x}{x^4}\,dx$$

$$\frac{x^3+\sin^2 x}{x^4} \geq \frac{x^3}{x^4} = \frac{1}{x}.$$

Since $\int_1^\infty \frac{1}{x}\,dx$ diverges, by the Comparison Theorem, so must $\int_1^\infty \frac{x^3+\sin^2 x}{x^4}\,dx$.

Question 10:

(a) [5 points] Determine the area of the region bounded between the curves $y = 4x - x^2$ and $y = x^2$.



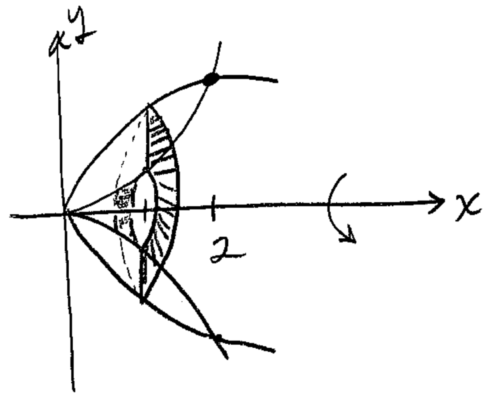
$$A = \int_0^2 4x - x^2 - x^2 \, dx$$

$$= \left[\frac{4x^2}{2} - \frac{2x^3}{3} \right]_0^2$$

$$= 8 - \frac{16}{3}$$

$$= \frac{24-16}{3} = \boxed{\frac{8}{3}}$$

(b) [5 points] The bounded region in part (a) is rotated about the x-axis. Determine the volume of the resulting solid. (Disks/washers would be best here.)



$$V = \int_0^2 \pi \left[(4x-x^2)^2 - (x^2)^2 \right] dx$$

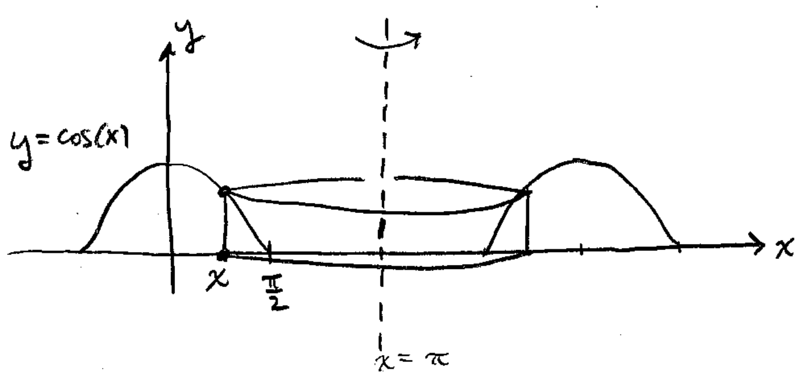
$$= \pi \int_0^2 \left[16x^2 - 8x^3 \cancel{+x^4} \cancel{-x^4} \right] dx$$

$$= \pi \left[\frac{16x^3}{3} - \frac{8x^4}{4} \right]_0^2$$

$$= \pi \left[\frac{128}{3} - 32 \right] = \pi \left[\frac{128-96}{3} \right] = \boxed{\pi \cdot \frac{32}{3}}$$

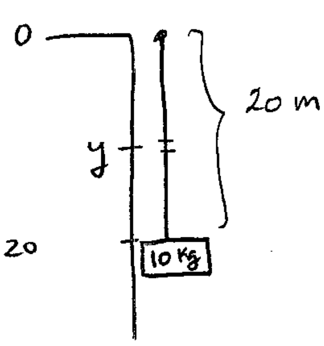
Question 11:

(a) [**4 points**] The curve $y = \cos(x)$, $-\pi/2 \le x \le \pi/2$, is rotated about the vertical line $x = \pi$. Set up but do not evaluate the integral representing the volume of the resulting solid. (Cylinders would be best here.)



$$V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\pi\,(\pi - x)\cos(x)\,dx$$

(b) [**4 points**] 20 m rope hangs over the side of a building and a 10 kg bucket is tied to the end of the rope. A person at the top of the building pulls the rope and bucket up onto the roof of the building. How much work is done if the rope has a total mass of 2 kg? Recall that the acceleration due to gravity is $g = 9.8$ m/s^2.



$$W_{\text{bucket}} = (10\,\text{kg})\left(9.8\,\frac{\text{m}}{\text{s}^2}\right)(20\,\text{m})$$

$$= 1960\ \text{Nm}$$

$$W_{\text{rope}} = \int_{y=0}^{20} \left(\frac{2\,\text{kg}}{20\,\text{m}}\right)(y\,\text{m})\left(9.8\,\frac{\text{m}}{\text{s}^2}\right)(dy\,\text{m})$$

$$= \frac{9.8}{10}\int_0^{20} y\,dy$$

$$= \frac{9.8}{10}\left[\frac{y^2}{2}\right]_0^{20}$$

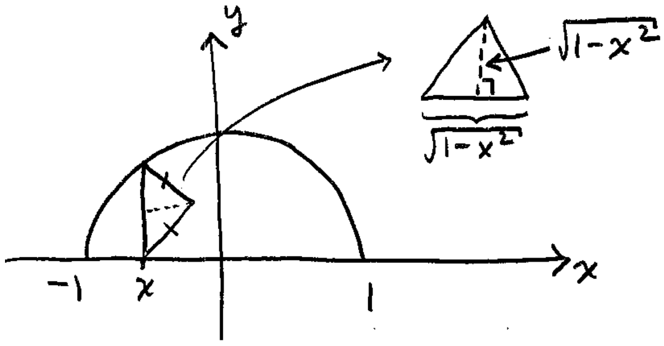
$$= \frac{9.8}{10}\cdot\frac{20}{2}\cdot 20$$

$$= 196\ \text{Nm}$$

$\therefore$ Total work is $W_{\text{bucket}} + W_{\text{rope}} = 1960 + 196 = \boxed{2156\ \text{Nm}}$

Question 12:

(a) [5 points] The base (flat bottom surface) of a solid is the region between the curve $y = \sqrt{1-x^2}$ and the x-axis. Note that $y = \sqrt{1-x^2}$ is the upper half of the circle of radius 1 and centre $(0, 0)$. Cross-sections perpendicular to the x-axis are isosceles triangles of equal height and base. Determine the volume of the solid.



$$\therefore A(x) = \frac{1}{2}\sqrt{1-x^2}\cdot\sqrt{1-x^2} = \frac{1-x^2}{2}.$$

$$\therefore V = \int_{-1}^{1} \frac{1-x^2}{2}\,dx = \frac{1}{2}\left[x - \frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{2}\left[\left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)\right] = \boxed{\frac{2}{3}}$$

(b) [5 points] Solve the following differential equation:

$$\frac{dy}{dx} = \frac{1+y^2}{\sqrt{x+1}}, \qquad y(0) = 1$$

You may leave your solution in implicit form (it is not necessary to isolate the y variable in your final answer.)

$$\int \frac{1}{1+y^2}\,dy = \int (x+1)^{-\frac{1}{2}}\,dx$$

$$\arctan(y) = 2(x+1)^{\frac{1}{2}} + C$$

$y(0) = 1$:

$$\arctan(1) = 2 + C$$

$$\therefore C = \frac{\pi}{4} - 2$$

$$\therefore \boxed{\arctan(y) = 2(x+1)^{\frac{1}{2}} + \frac{\pi}{4} - 2.}$$

Question 13:

(a)[3 points] Determine the first three non-zero terms of the Maclaurin series for the function $f(x) = x^3 e^{x^2}$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

$$x^3 e^{x^2} = x^3 + x^5 + \frac{x^7}{2!} + \cdots$$

$\therefore$ first three non-zero terms are $x^3 + x^5 + \frac{x^7}{2!}$

(b)[4 points] Use a Maclaurin series (not L'Hospital's Rule) to evaluate the limit

$$\lim_{x\to 0} \frac{1-\cos(x^2)}{x^3(e^x - 1)}$$

$$= \lim_{x\to 0} \frac{1 - \left[1 - \frac{x^4}{2!} + \frac{x^8}{3!} - \cdots\right]}{x^3\left[\cancel{1} + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \cancel{-1}\right]}$$

$$= \lim_{x\to 0} \frac{\frac{x^4}{2!} - \frac{x^8}{3!} + \cdots}{x^4 + \frac{x^5}{2!} + \frac{x^6}{3!}}$$

$$= \lim_{x\to 0} \frac{\frac{1}{2!} - \frac{x^4}{3!} + \cdots}{1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots}$$

$$= \boxed{\frac{1}{2}}$$

(c)[3 points] Determine the Maclaurin polynomial of degree three for $f(x) = e^x \sin x$. You may use the definition or any other valid method to obtain your answer.

$$f(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

$$= x + x^2 - \frac{x^3}{3!} + \frac{x^3}{2!} + \left(\text{terms of degree greater than } 3\right)$$

$$= x + x^2 + \frac{3x^3 - x^3}{6} + \left(\text{terms of degree greater than } 3\right)$$

$$\therefore \boxed{T_3(x) = x + x^2 + \frac{1}{3}x^3}.$$