Question 1:

(a) Use the midpoint rule on 2 subintervals to approximate $\int_0^{\pi} \sin(x)\cos(x)\,dx$. Simplify your final answer.

x: $0 \quad \frac{\pi}{4} \quad \frac{\pi}{2} \quad \frac{3\pi}{4} \quad \pi$ $\Big\}$ $\bar{x}_1 = \frac{\pi}{4}, \; \bar{x}_2 = \frac{3\pi}{4}, \; \Delta x = \frac{\pi}{2}$

$f(x) = \sin(x)\cos(x)$; $f(\bar{x}_1) = (\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}}) = \frac{1}{2}$; $f(\bar{x}_2) = (\frac{1}{\sqrt{2}})(\frac{-1}{\sqrt{2}}) = \frac{-1}{2}$

$$\therefore M_2 = \Delta x \left[f(\bar{x}_1) + f(\bar{x}_2) \right]$$

$$= \frac{\pi}{2}\left[\frac{1}{2} + \frac{-1}{2}\right]$$

$$= \boxed{0}$$

[5]

(b) Give an error bound on your approximation in part (a). Simplify your final answer.

$f(x) = \sin(x)\cos(x)$

$f'(x) = \cos(x)\cos(x) - \sin(x)\sin(x) = \cos^2(x) - \sin^2(x)$

$f''(x) = -2\cos(x)\sin(x) - 2\sin(x)\cos(x) = -4\sin(x)\cos(x)$

$|f''(x)| = |-4\sin(x)\cos(x)| \le 4$ on $[0,\pi]$, so $K=4$

$$\therefore |E_{M_2}| \le \frac{K(b-a)^3}{24n^2} = \frac{4(\pi-0)^3}{24 \cdot 2^2} = \boxed{\frac{\pi^3}{24}}$$

Alternatively:

$|f''(x)| = |-4\sin(x)\cos(x)|$

$= |-2\sin(2x)|$

≤ 2 on $[0,\pi]$, so $K=2$,

in which case $|E_{M_2}| \le \frac{\pi^3}{48}$

[5]

Question 2:

(a) Determine $\int_0^\infty xe^{-x^2}\,dx$ making proper use of any required limits.

$$\int_0^\infty xe^{-x^2}dx = \lim_{b\to\infty}\int_0^b xe^{-x^2}dx$$

let $u=-x^2$

$du=-2x\,dx$

$$= \lim_{b\to\infty}\left[\frac{e^{-x^2}}{-2}\right]_0^b$$

$$= \lim_{b\to\infty}\underbrace{\frac{e^{-b^2}}{-2}}_{\to 0}-\left(\frac{e^{-0^2}}{-2}\right)$$

$$= \boxed{\frac{1}{2}}$$

[5]

(b) Your answer in part (a) represents the area in the first quadrant between $y = xe^{-x^2}$ and the x-axis. Determine the value of q such that half of the area lies to the left of the line $x = q$.

$$\int_0^q xe^{-x^2}dx = \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)$$

$$\left[\frac{e^{-x^2}}{-2}\right]_0^q = \frac{1}{4}$$

$$\frac{e^{-q^2}}{-2}-\left(\frac{e^{-0^2}}{-2}\right)=\frac{1}{4}$$

$$\frac{e^{-q^2}}{-2}+\frac{1}{2}=\frac{1}{4}$$

$$\frac{e^{-q^2}}{-2}=-\frac{1}{4}$$

$$e^{-q^2}=\frac{1}{2}$$

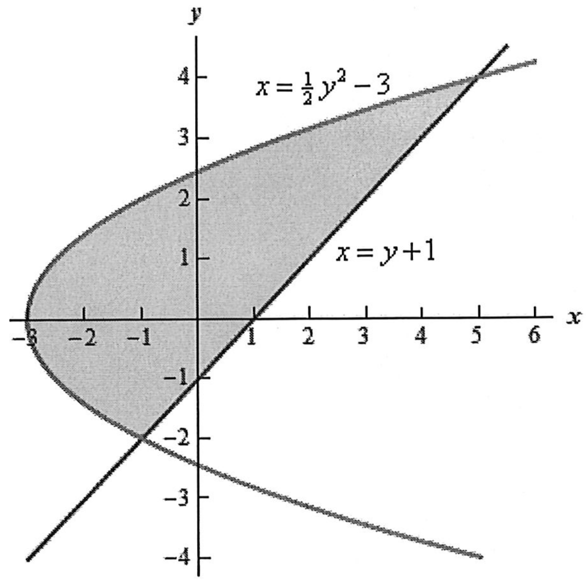
$$-q^2=\ln(1/2)$$

$$q^2=-\ln\left(\tfrac{1}{2}\right)=\ln(2)$$

$$\boxed{q=\sqrt{\ln(2)}}$$

[5]

Question 3: Determine the area of the shaded region:



$$A = \int_{y=-2}^{y=4} (y+1) - (\tfrac{1}{2}y^2 - 3)\, dy$$

$$= \int_{-2}^{4} -\tfrac{1}{2}y^2 + y + 4\, dy$$

$$= -\tfrac{1}{6}\left[y^3\right]_{-2}^{4} + \tfrac{1}{2}\left[y^2\right]_{-2}^{4} + 4\left[y\right]_{-2}^{4}$$

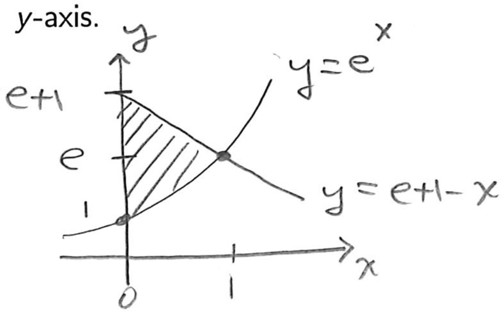
$$= -\tfrac{1}{6}(64+8) + \tfrac{1}{2}(16-4) + 4(4+2)$$

$$= -12 + 6 + 24$$

$$= \boxed{18}$$

[5]

Question 4: Determine the area of the region in the first quadrant bounded by $y = e^x$, $y = e + 1 - x$ and the y-axis.



$$A = \int_{x=0}^{1} (e+1-x) - e^x\, dx$$

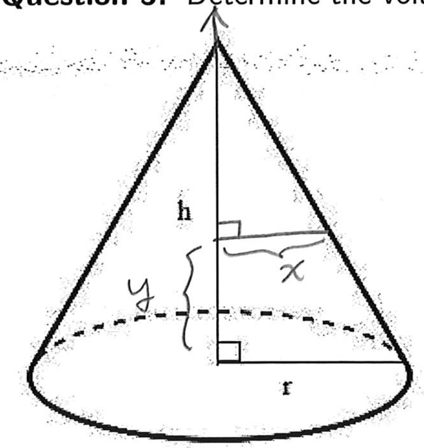
$$= \left[(e+1)x - \frac{x^2}{2} - e^x\right]_0^1$$

$$= \left[\cancel{e}+1 - \tfrac{1}{2} - \cancel{e}\right] - \left[0 - 0 - 1\right]$$

$$= \boxed{\frac{3}{2}}$$

[5]

Question 5: Determine the volume of a cone of base radius r and height h.



$$\frac{h-y}{x} = \frac{h}{r} \Rightarrow x = \frac{r}{h}(h-y).$$

$$V = \int_{y=0}^{h} \pi \left[\frac{r}{h}(h-y)\right]^2 dy$$

$$= \frac{\pi r^2}{h^2} \int_0^h (h-y)^2\, dy$$

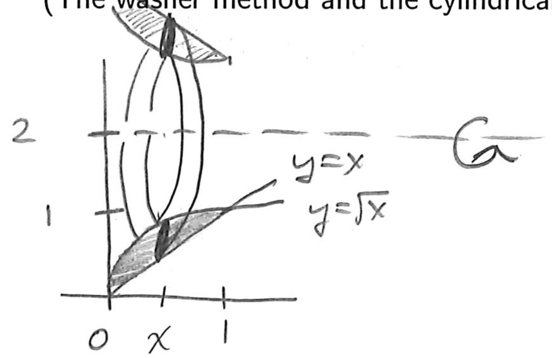
$$= \frac{\pi r^2}{h^2} \cdot \frac{-1}{3} \left[(h-y)^3\right]_0^h$$

$$= \frac{\pi r^2}{h^2} \cdot \frac{-1}{3} \cdot \left[0 - h^3\right]$$

$$= \boxed{\frac{1}{3}\pi r^2 h}$$

[5]

Question 6: The region in the first quadrant bounded between $y = x$ and $y = \sqrt{x}$ is rotated about the line $y = 2$. Determine the volume of the resulting solid.
(The washer method and the cylindrical shell method are both equally effective for this question.)



Washers: $V = \int_{x=0}^{1} \pi (2-x)^2 - \pi (2-\sqrt{x})^2\, dx$

$$= \pi \int_0^1 \cancel{4} - 4x + x^2 \cancel{-4} + 4x^{1/2} - x\, dx$$

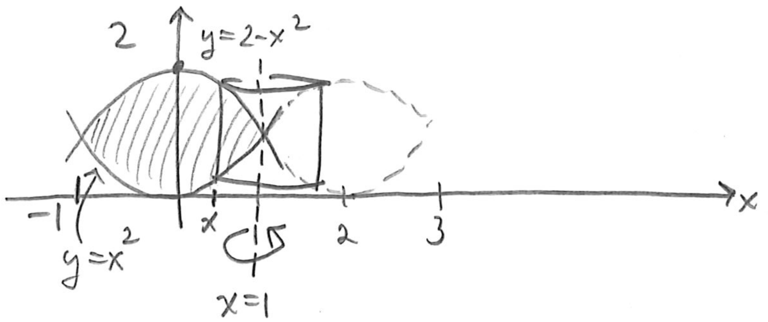
$$= \pi \int_0^1 x^2 - 5x + 4x^{1/2}\, dx$$

$$= \pi \left[\frac{x^3}{3} - \frac{5x^2}{2} + \frac{8}{3}x^{3/2}\right]_0^1$$

$$= \boxed{\frac{\pi}{2}}$$

[5]

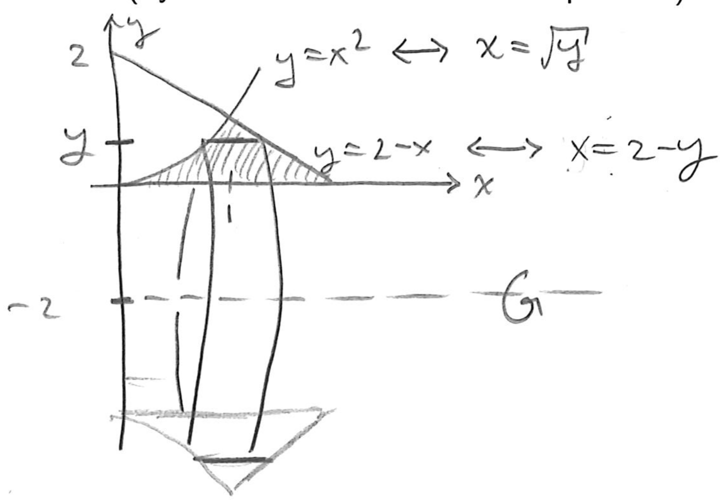
Question 7: The region bounded between $y = x^2$ and $y = 2 - x^2$ is rotated about the line $x = 1$. Set up BUT DO NOT EVALUATE the integral representing the volume of the resulting solid. (Cylindrical shells is best for this question.)



$$\boxed{V = \int_{x=-1}^{1} 2\pi (1-x)\left(2-x^2-x^2\right) dx}$$

[5]

Question 8: The region in the first quadrant bounded between $y = x^2$, $y = 2 - x$ and the x-axis is rotated about the line $y = -2$. Set up BUT DO NOT EVALUATE the integral representing the volume of the resulting solid. (Cylindrical shells is best for this question.)



$$\boxed{V = \int_{y=0}^{1} 2\pi (y+2)\left(2-y-\sqrt{y}\right) dy}$$

[5]